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Approaches to the Four Colour Theorem

A Thesis Submitted for the Degree of Doctor of Philosophy

Faculty of Mathematics

The Open University

October 1992

Author number: M709845X
Date of submission: 27 October 1992
Date of award: 25 April 1995

APPROACHES TO THE FOUR COLOUR THEOREM

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CHAPTER ONE

Introduction

In attempting to prove or disprove the notion that all planar maps are Guthrie colourable (i.e. face 4-colourable) we need only look at cubic maps. Suppose that some map having a vertex of degree, say, seven was not 4 colourable (Fig 1.1). Then we could inflate the map at that vertex with a heptagon. Then the new, cubic, map is not four colourable if the original map was not.

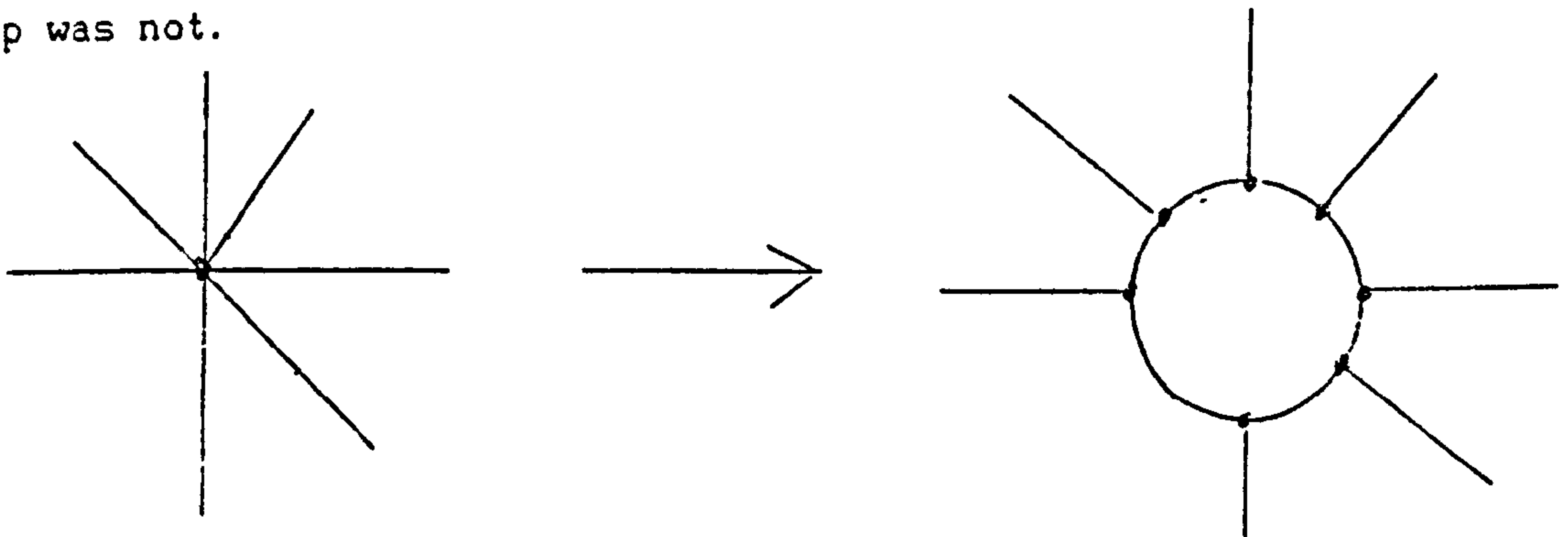


Fig. 1.1

That four colours are needed is shown by Fig.1.2. It follows quickly from Euler's formula that any bridgeless cubic map has at least one face of order less than 6. That the 5-colouring of a map can be made to depend on that of a simpler map is readily apparent in cases when the order of the smallest face is 2,3 or 4. In the case when the smallest face is a pentagon, we can either use Kempe chains or the following more direct approach.

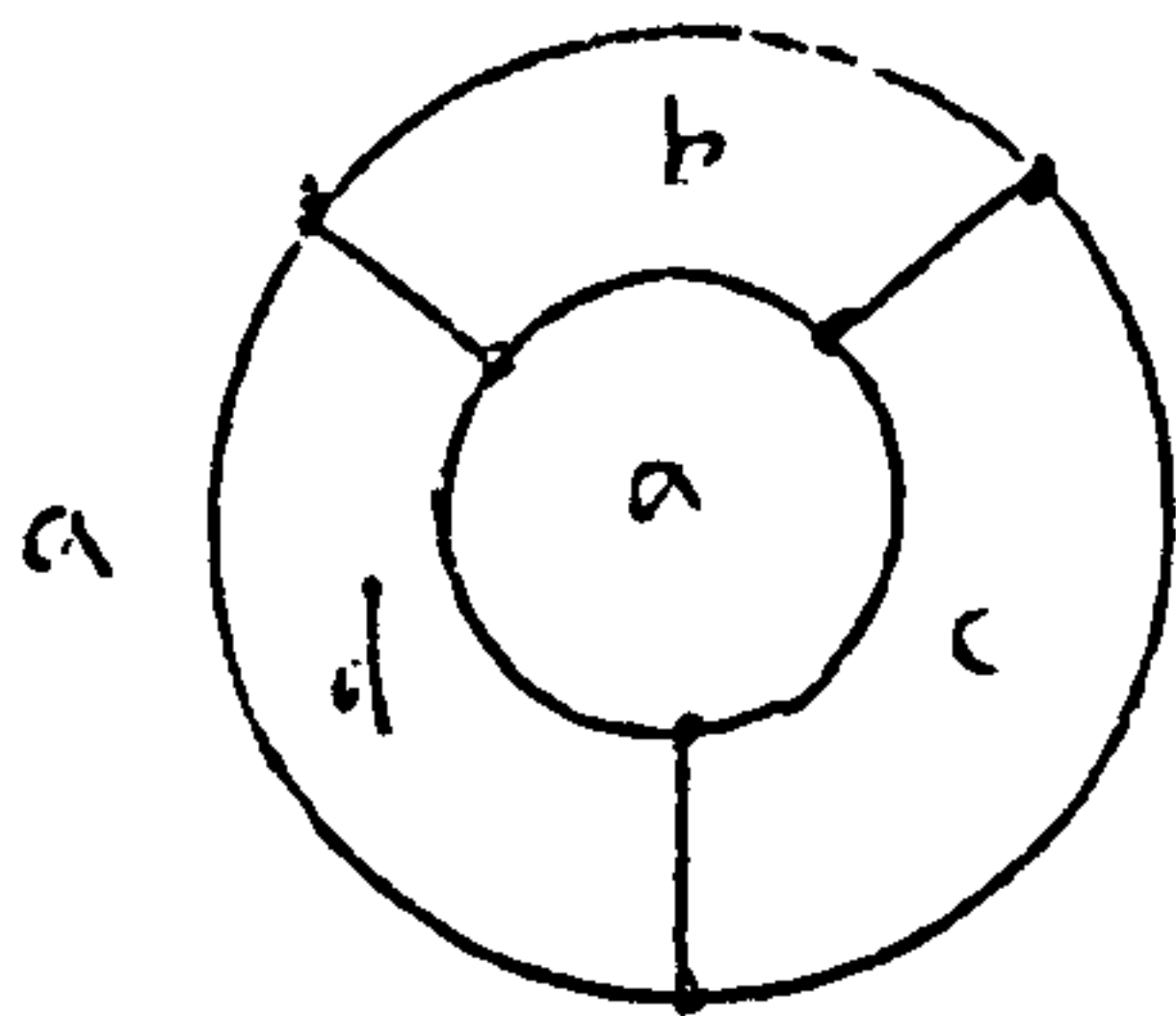


Fig. 1.2

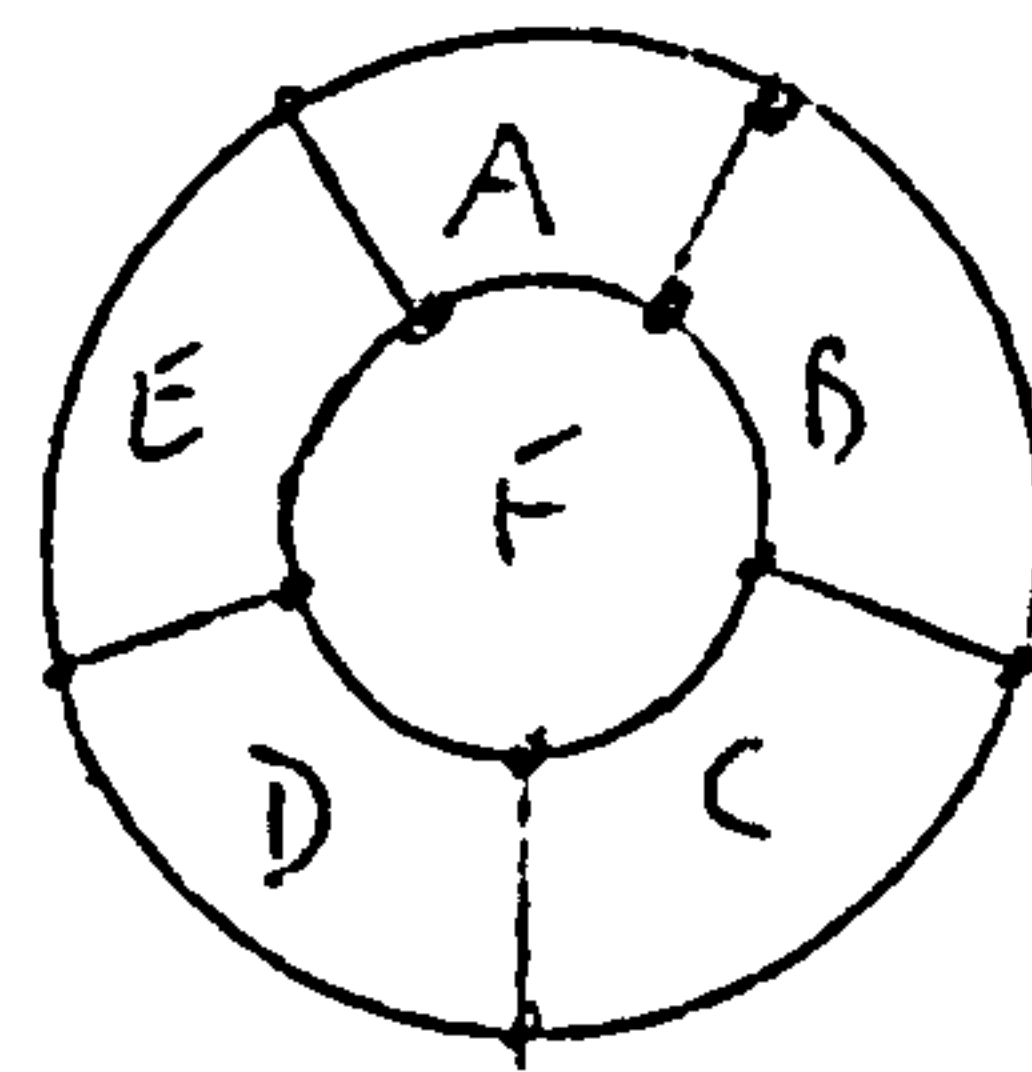


Fig. 1.3

Let F be a pentagon adjacent to faces A, B, C, D, E in a connected map.

(Fig.1.3.) Then given the connectedness of the plane, we can say that at most one of the following numbered statements is true:

- (1) B and D are the same face.
- (2) B and D are adjacent faces.
- (3) C and E are the same face.
- (4) C and E are adjacent faces.

So in particular we can choose either $(B \text{ and } D)$ or $(C \text{ and } E)$ as not having any numbered statement applying to them. Say the pair is $(B \text{ and } D)$. So we can join B and D to form a smaller map on whose 5 colouring the colouring of the given map can now readily be seen to depend. (Fig.1.4) If the map becomes disconnected, we deal with each part separately.

A colouring of a graph can be seen as two quite different things. This dichotomy is equally apparent in most of the reformulations of Guthrie colourability and is almost endemic to it. First there is the question of assigning elements of a group to a graph so a certain system of inequalities holds. Then we have the problem of assigning group elements so that a system of equations within the group has roots all lying within some specified subset of the group. Very often in this latter case, the group may be made into a field by the addition of a multiplicative group.

In this latter case the colouring set has the structure of a field, which need not be present in the former cases (i.e. when solving inequalities; and the case of restricted equalities when the group cannot be made into a field.)

Summary of Chapter 2

In this chapter we examine a list of properties a graph may have

from almost trivial 2-colourings up to very structured graphs which have 4-colourings with structure in the Klein 4-group having a certain order around both vertices and faces (See Fig 1.4).

Each of these properties implies the graph has all the preceding ones.

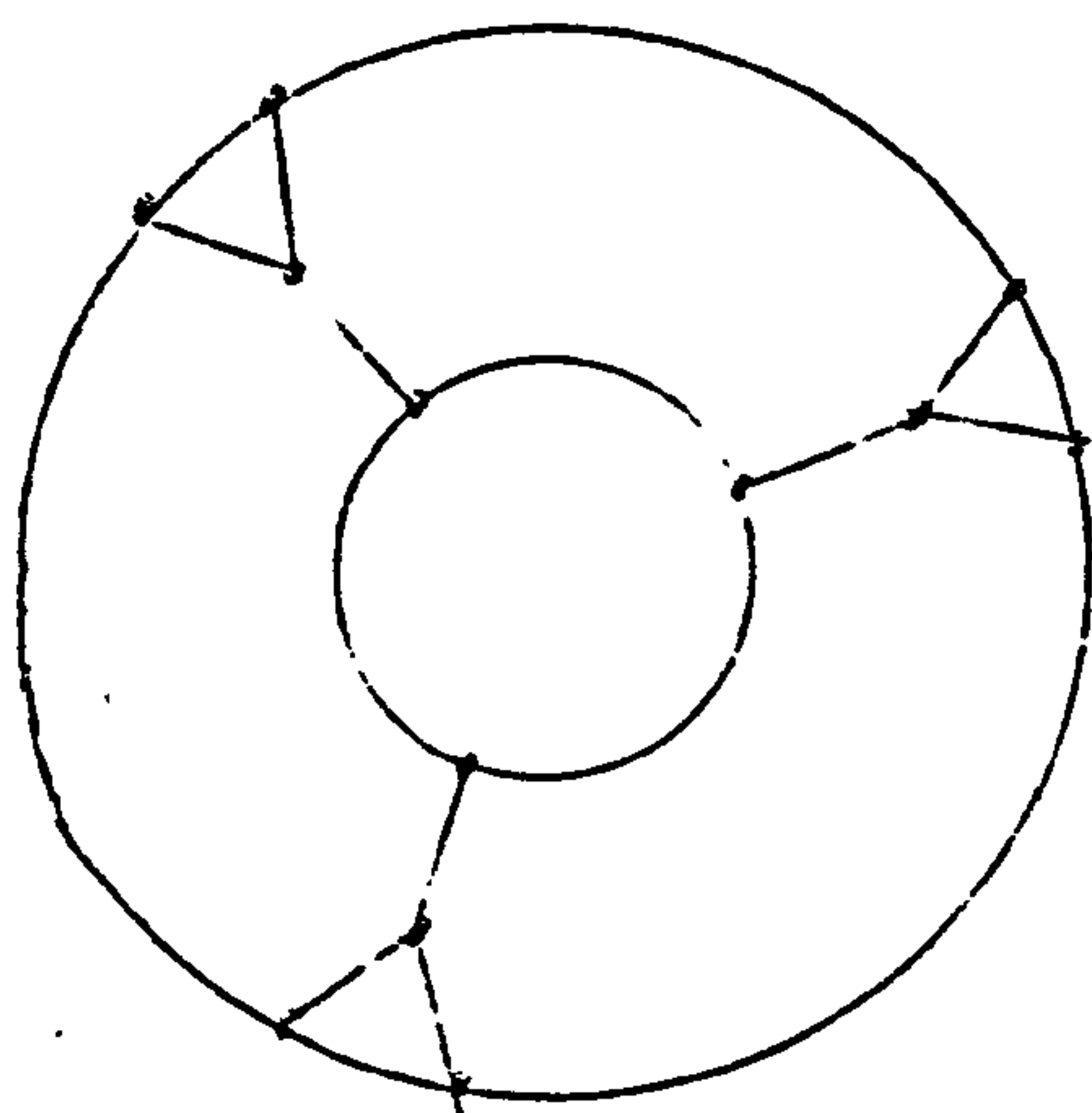


Fig. 1.4

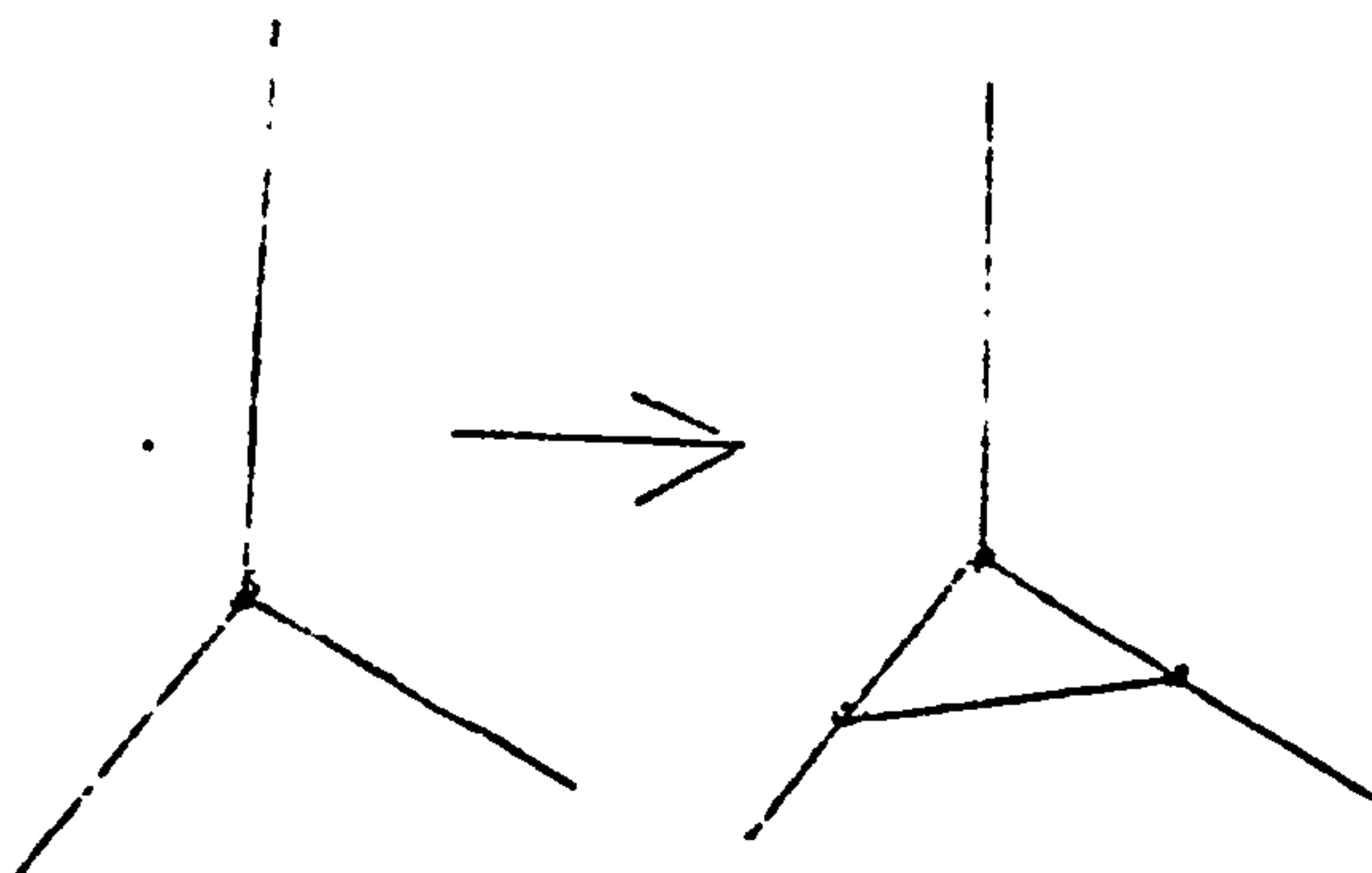


Fig. 1.5

By the *inflation* of a cubic vertex we mean that a triangle has been joined into the vertex (See Fig 1.5). We shall see that the four colour theorem implies that all plane graphs have inflations with all the other properties listed. Thus four colourability is a strong asset a plane graph may have, and comes immediately after the last in the list for which plane counterexamples have been found.

The rest of Chapter two deals with an already known result concerning labeling angles with a restricted subset of Z_4 in which certain equalities may hold.

Summary of Chapter 3

This chapter deals exclusively with snarks. We see that Rufus Isaacs was instrumental in developing the theory surrounding what are now known as "Loupekine's Snarks". We present an approach to these snarks different from

Isaacs, which is more general as it suggests many more ways of forming new but imprime snarks.

Summary of Chapter 4

P.J.Heawood gave his name to the vertex labellings with a restricted subset of Z_3 , and he firmly believed that the four colour problem would yield to the algebra and all that is known about this simple field. He went on to discover many cases of un-colourability, and we humbly offer up one more such case of un-colourability. (Or should one say un-labelability ?)

To deal with the radically different work in Chapter 5 we need a lemma in Z_3 . So we develop fully the angle labelling theories. The results we discover are original and, hopefully, interesting in their own right.

Summary of Chapter 5

We denote by K the Klein 4 - group. If we take the difference in K between labels of faces in a plane cubic map, we discover the equivalence of Tait to Guthrie labelling. Then if we change group and take differences in Z_3 of Tait labels, we discover Heawood's labels. Then if we take differences between Heawood labels we discover the S_3 labels that are the subject of this chapter.

We might expect to go round in circles extracting new labellings from old. But if we proceed beyond what we do in Chapter 5 we get deep in non Abelian algebra. We leave the groups Z_4 and Z_5 for another day. We feel these are suitable subjects for further investigation.

Summary of Chapter 6

Although we have dealt exclusively with K in this chapter, the same matter

can be gone over in any field of character two, with similar results.

However the Klein group has four elements so it is natural to pick out this work in a study of the four colour theorem. The other very interesting case, that of two elements, is dealt with in Chapter 2, and we return to it in the final chapter.

We define a Klein labelling of the interchange graph of a plane map M and go on to showing that it corresponds to both a face 4-colouring and a vertex 4-colouring simultaneously. We will pick up the thread in Chapter 8.

Summary of Chapter 7

The notion of *sample* is a new one when applied to graph colouring. We see how the chromatic number of a graph satisfies

$$\chi \leq \prod \chi_i$$

where the χ_i are the chromatic numbers of a number of subgraphs making up M . In the case that a spanning part of the edge set $E(M)$ can be broken into even cycles, this formula yields chromatic number four. In this case one set of the cycles is an *Even Component Generator*, or what is the same thing, an odd cut sample.

Summary of Chapter 8

In this final chapter we use the theory on Kleinian labels to return to a binary labelling of the angles of the interchange graph. The structure imposed by such a $(0,1)$ -labelling is the very least needed to construct the original 4-colourings of faces and vertices, and many of such 4-colourings correspond to each so called *Alternate Angle Labelling*.

CHAPTER TWO

Labelling With Elements of Finite Groups and Fields

Classical mathematics is the mathematics of equalities. Graph colouring theory is one of the first extensive studies of inequalities. It is remarkable that this theory is so dense with unsolved problems for just as few colours as three or four. Given a colouring problem, often we can relate it to some other problem -- a set of equalities in a group or field. But then we find that for a solution to have any meaning in relation to the problem first formulated, the solution space is limited. The transition from sets of inequalities in one mathematical structure to restricted equalities in another and vice versa is what this exposition is all about. The object of study is the famous four colour theorem for planar graphs.

Some properties of a graph imply others. In particular, the existence of a k -colouring implies a that of a $(k+1)$ -colouring (unless the number of elements to be coloured is k). In this chapter we shall see how some simple labelling properties of a cubic graph are nested in this sense, the four colour property appearing in the middle. Thus the last in the list implies all the others, and the four colour property can be modified to imply all the others, in a sense.

But first we need some elementary concepts.

For the basic concepts of graph theory, we refer to [Wilson]. In particular, we recall that a graph G may be defined as a vertex set $V(G)$ together with an edge set $E(G)$ of unordered pairs $\{u, v\}$ of elements of

$V(G)$. We also recall that a *hypergraph* is a vertex set $V(G)$ together with a set of unordered subsets of 2 or more elements of $V(G)$.

We now supplement these definitions as follows.

Definitions

By a *hypograph*, G , we mean a vertex set $V(G)$ together with a list $E(G)$ of subsets of $V(G)$, each of size 1 or 2, the *edges*. *Incidence* is defined as for graphs, as is the concept of the *degree* of a vertex. Edges of size 1 are *dangling edges*; those of size 2 are *normal edges*. We allow a vertex to be incident with more than one dangling edge (this is why we speak of a list rather than a set of subsets), but we do not allow multiple normal edges.

Definitions of the elementary groups we shall encounter

(1) The cyclic groups, $C_n = \{0, 1, \dots, n-1\}$ under the operation of addition modulo n .

We shall use C_2 , C_3 and C_4 in this thesis; in particular, much of this chapter is concerned with C_2 .

(2) The Klein 4-group, $K = \{a, b, c, d\}$, with d denoting the identity; this group is isomorphic with $C_2 \times C_2$.

(3) The symmetric group on three elements:

$$S_3 = \langle a, b: a^3 = b^2 = d, a^2b = ba \rangle,$$

with d again denoting the identity. (This is in order to reserve the symbol e to denote an edge of a graph.)

A plane bridgeless graph is often referred to as a *map*. As it appears so often, a plane cubic bridgeless graph will be referred to by its initial letters *p.c.b.*

Further definitions

If G is a graph or hypograph drawn in the plane without crossings, then in addition to $V(G)$ and $E(G)$, we have the set of *faces* of G , $F(G)$, that is, the set of connected components of the complement in the plane of the drawing of G . We also have the set of *angles* of G , $A(G)$, which may be considered as intersections of vertex neighbourhoods with faces. Finally, we have the set of *sides of edges* of G , $SE(G)$, each edge having a side in each of two faces. In an obvious way, there are *adjacency* relations on $F(G)$ and on $A(G)$, and *incidence* relations between each pair of sets $V(G)$, $E(G)$, $F(G)$, $A(G)$, $SE(G)$.

The *order* of a face of a map is the number of edges with which it is incident.

A *face-even* map is one in which each face has even order. See Fig 2.1.

A *vertex-even* graph is one in which each vertex has even degree. See Fig 2.2.

A *triple* map is one in which the order of each face is a multiple of 3. See Fig 2.3.

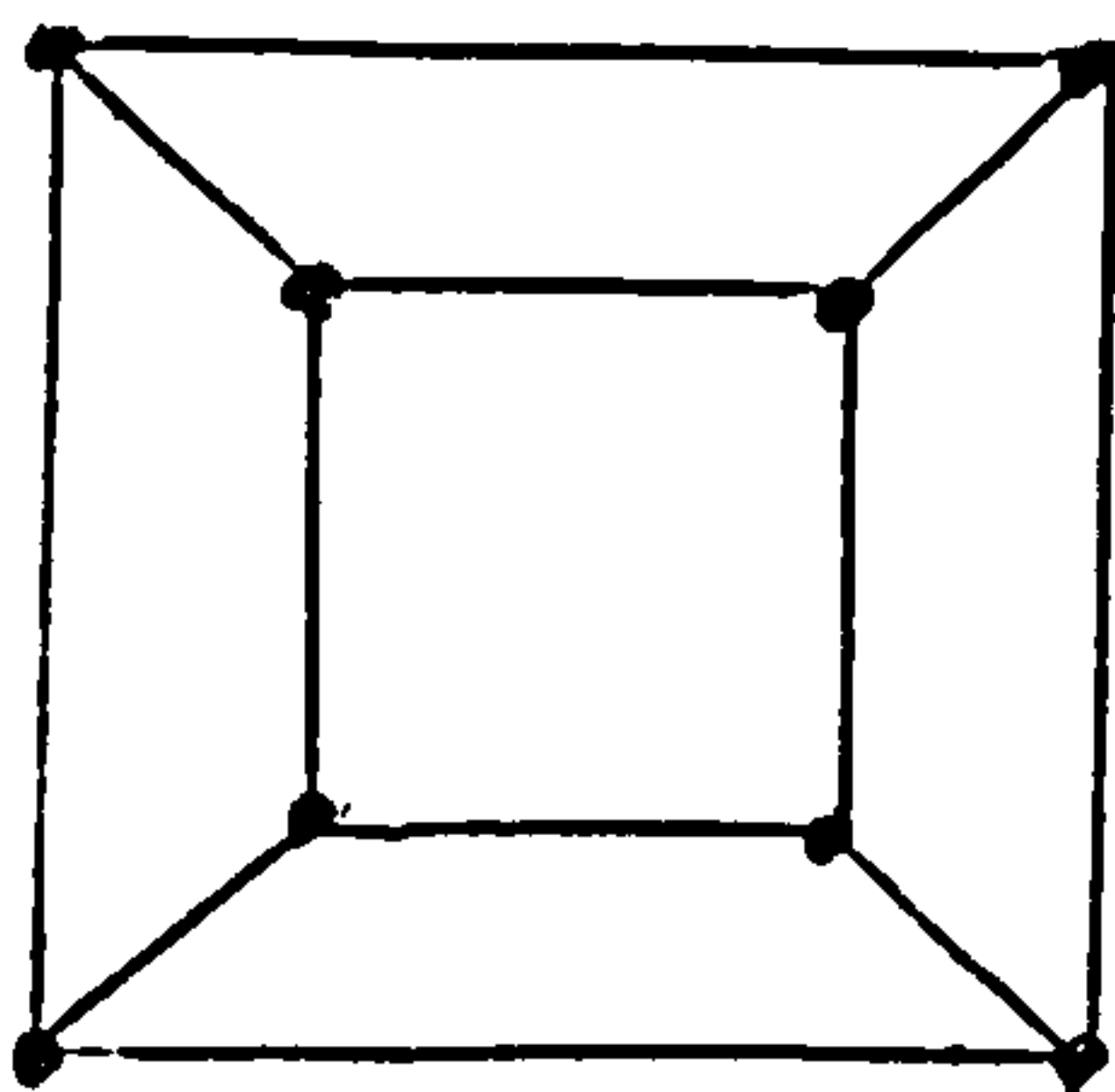


Fig 2.1

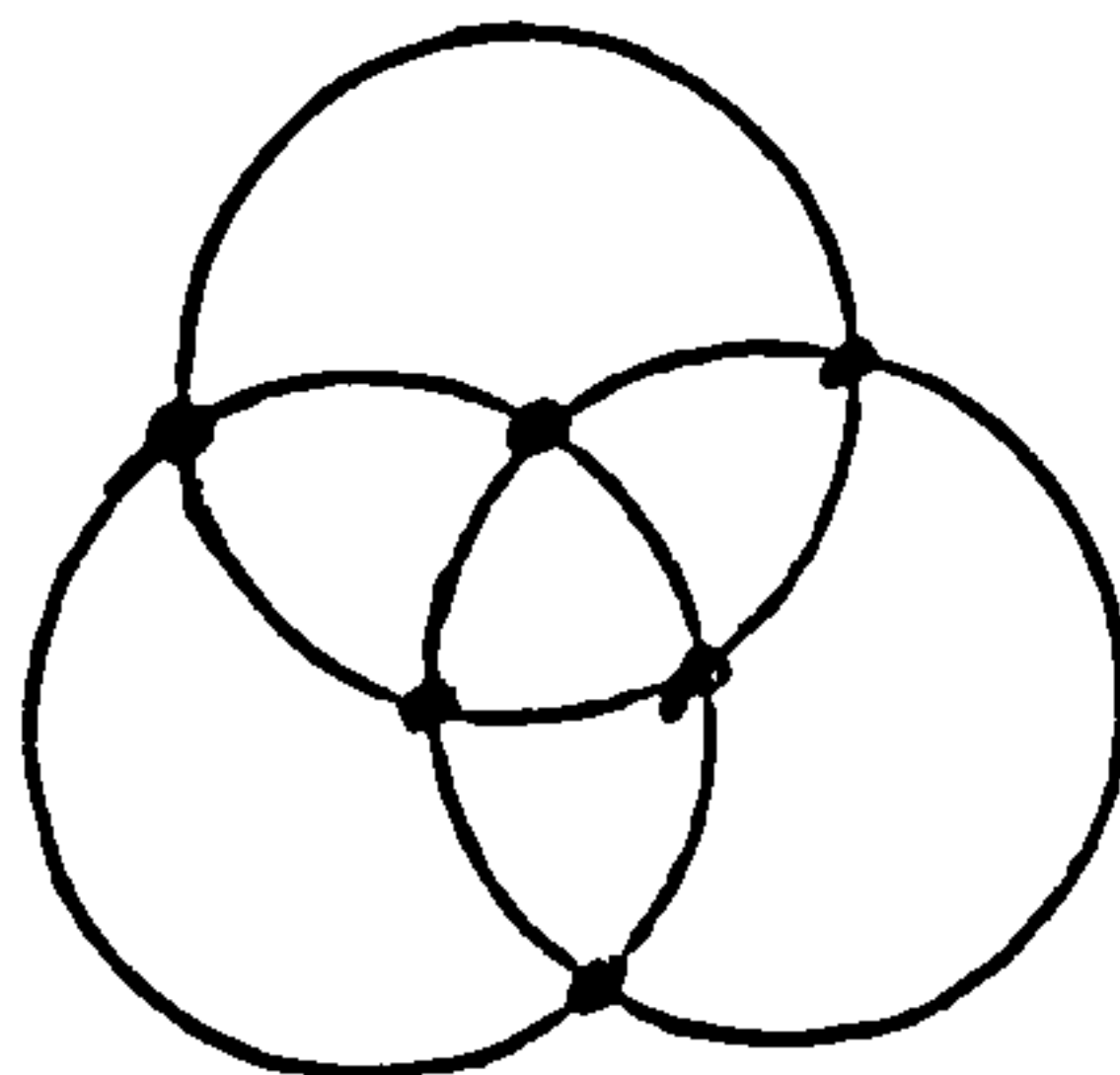


Fig 2.2

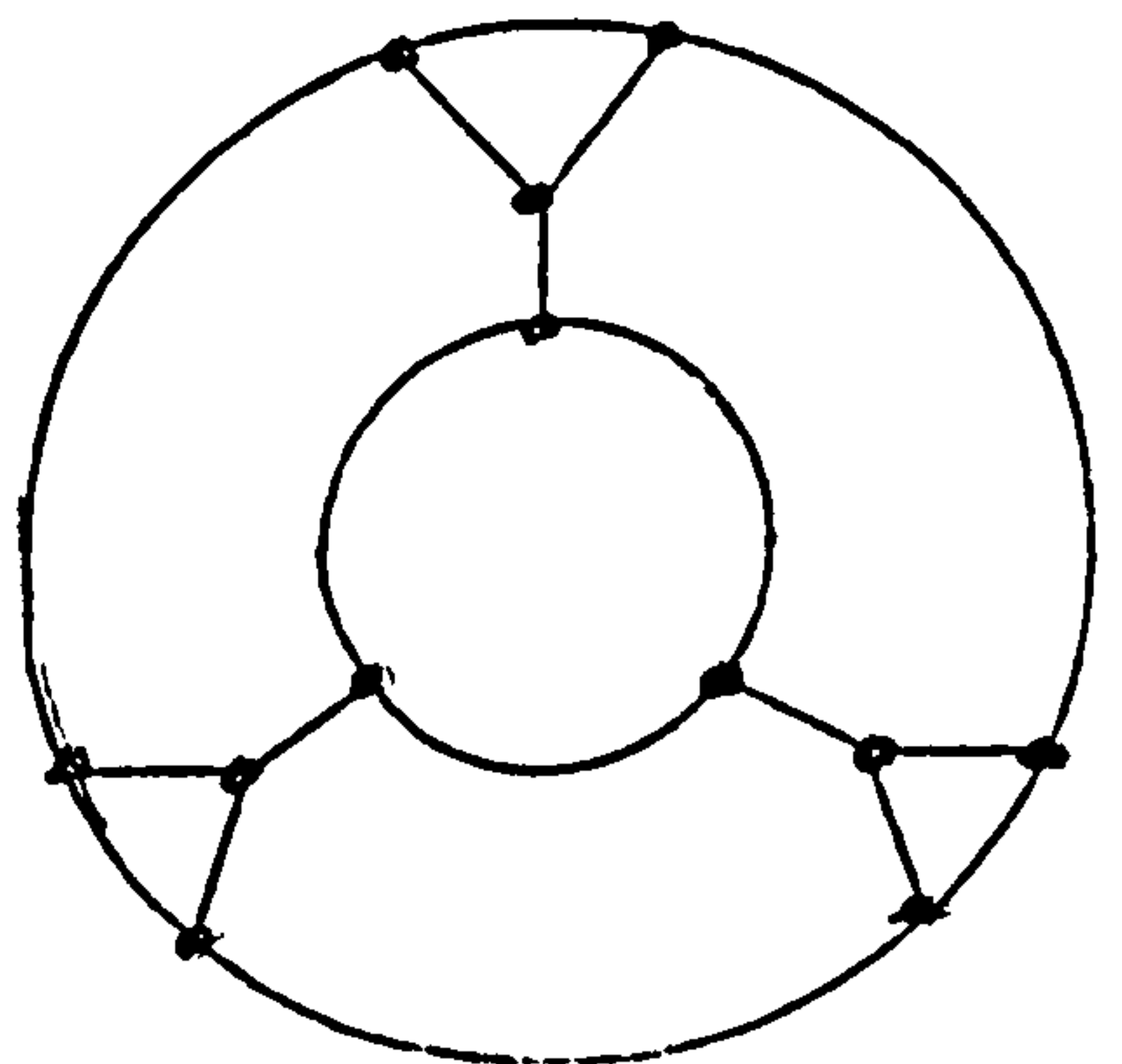


Fig 2.3

A *labelling* of $V(G)$, $E(G)$, $F(G)$, $A(G)$ or $S(G)$ is a function from the set in question to some other set, usually one of the groups above or a subset thereof. A vertex [resp. edge, face] *colouring* is a labelling in

which adjacent vertices [resp. edges, faces] have distinct labels. The *chromatic number* [resp. *index*] of G is the least number of colours required in a vertex [resp. edge] colouring of G , and is denoted by $\chi(G)$ [resp. $\chi'(G)$].

A *Guthrie colouring* of a map is a face 4-colouring; a *Tait colouring* of a cubic graph is an edge 3-colouring. We normally use the four elements of K for Guthrie colourings and the three non-identity elements a, b, c of K for Tait colourings.

The Parity Lemma for Regular Hypographs

The following theorem is a generalisation of the parity lemma for graphs [Descartes, Chetwynd].

Theorem 2.1

Let G be a k -regular hypograph. In any edge k -colouring of G , if there are n_i dangling edges of colour i ($i = 1, 2, \dots, k$), then

$$n_1 \equiv n_2 \equiv \dots \equiv n_k \equiv |V(G)| \pmod{2}. \quad (\text{See Fig 2.4.})$$

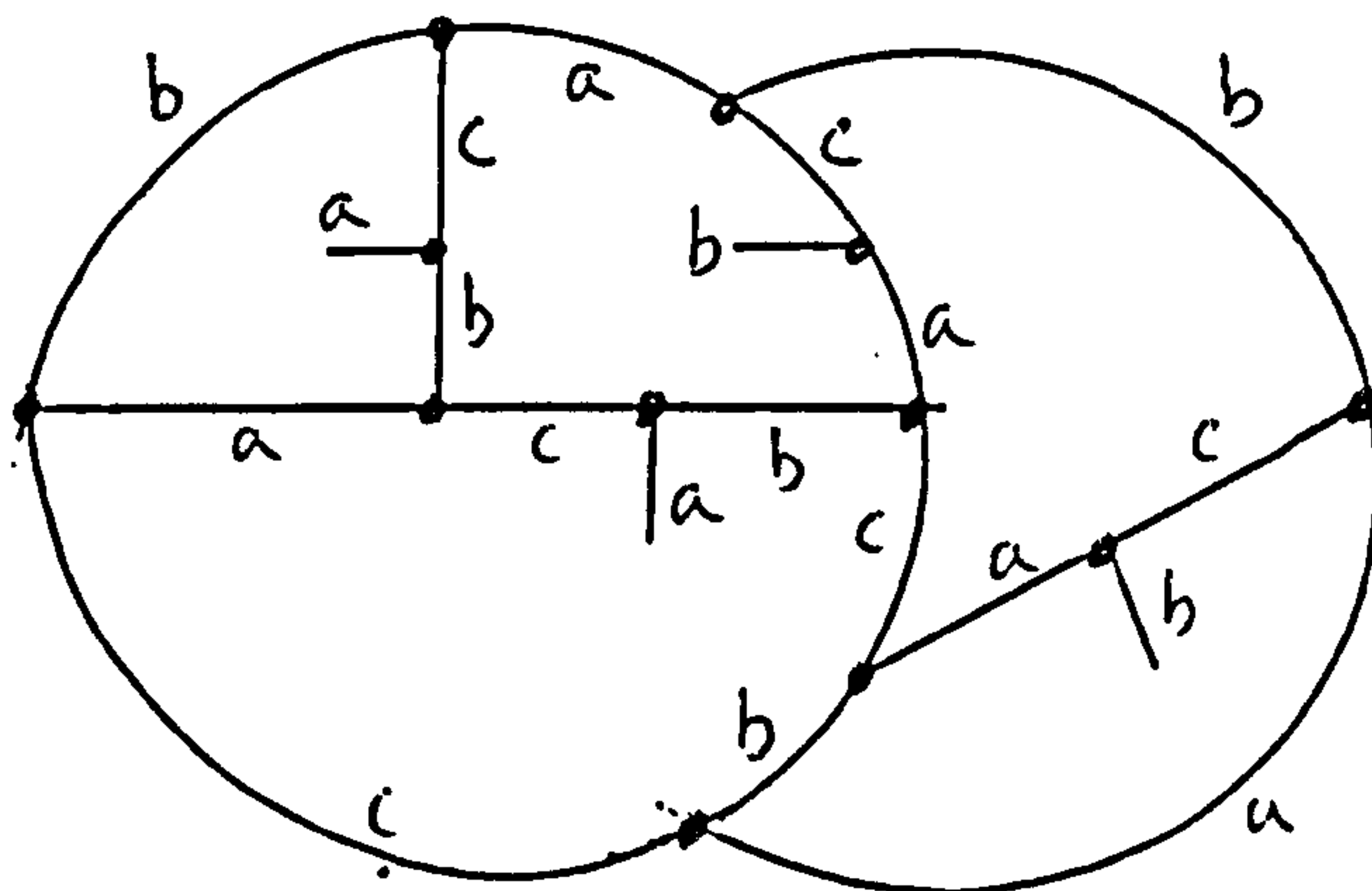


Fig 2.4

Proof

Let there be m_i normal edges of colour i ($i = 1, 2, \dots, k$). Then, as each colour appears once at each vertex,

$$n_i + 2m_i = |V(G)| \quad (i = 1, 2, \dots, k),$$

and the result follows immediately. ■

Corollary 2.1

Let G be a cubic graph that has been edge 3-coloured with colours 1, 2 and 3. If a cut splits n_i edges of colour i , for $i = 1, 2, 3$, then

$$n_1 \equiv n_2 \equiv n_3 \equiv N \pmod{2}$$

where N is the total number of edges cut. See Fig 2.5.

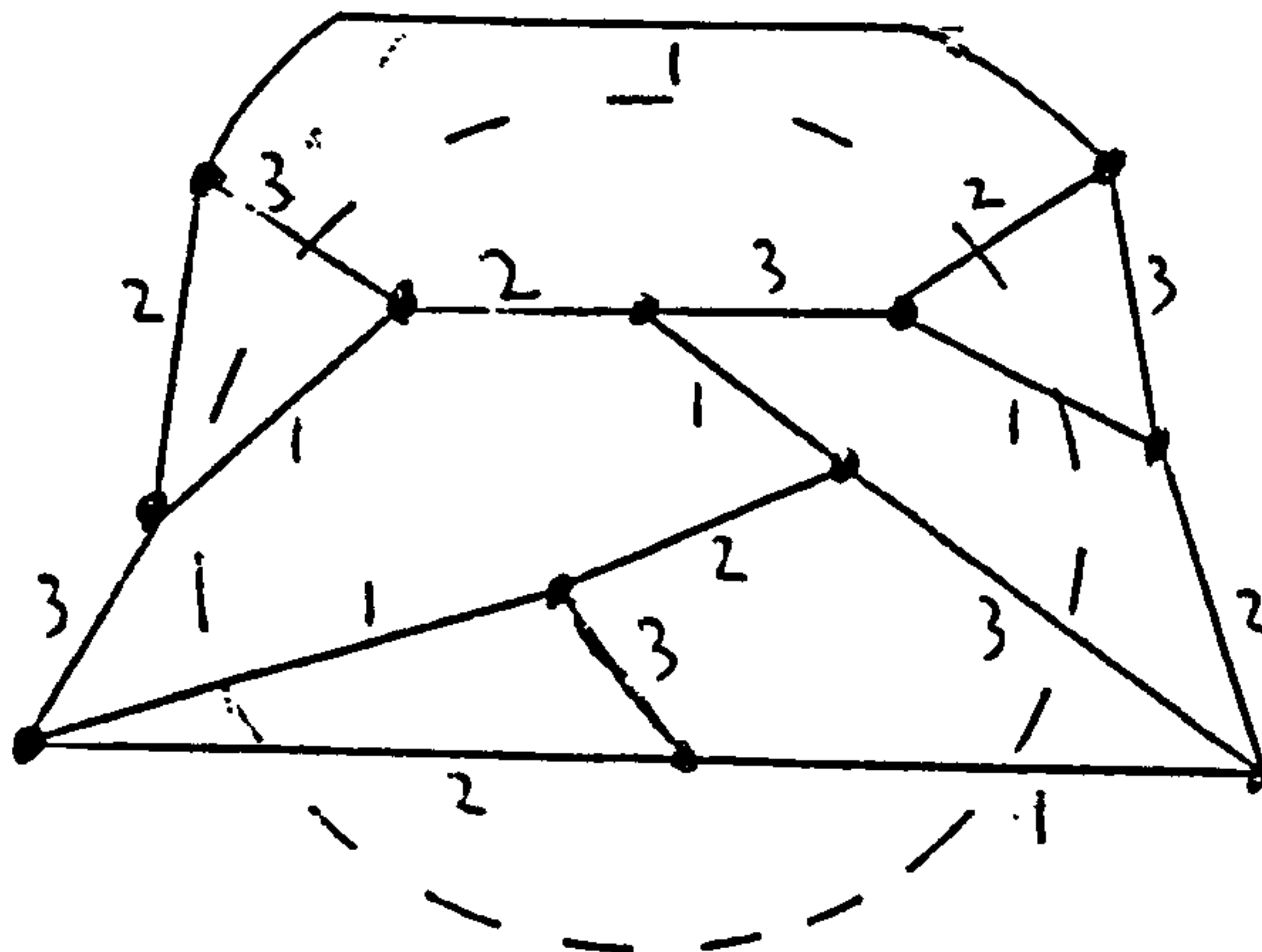


Fig 2.5

Now for a generalisation of Corollary 6.5 of [Florini and Wilson].

Theorem 2.2

If Graph G is regular of degree R except for one vertex of degree $r < R$, then it is of class 2.

Proof

Suppose G can be edge R -coloured. Then no pair of edges at the vertex of degree r can take the same colour. If we draw a cut round this vertex, and apply Theorem 2.1 to the dangling edges, then we see that if any colours are present they must all be present - just once. This contradicts the statement that $r < R$. ■

Conjecture A

Any planar 3-regular bridgeless hypograph, having $k \neq 1$ dangling edges, can be Tait coloured.

This is a difficult conjecture, and in the case of $k = 0$ it is equivalent to Tait's form of the four colour theorem (see Chapter Three). The case $k = 1$ is immediately excluded by Theorem 2.1. In case $k = 2$, the conjecture is equivalent to the Tait colourability of any bridgeless cubic graph which is planar except possibly for one edge, which may cross an unlimited number of other edges. (Cutting the offending edge produces a hypograph with two dangling edges, which by the Parity Lemma must have the same colour in any Tait colouring.) By a similar argument, in case $k = 3$ the conjecture is equivalent to the Tait colourability of any bridgeless cubic graph which is planar except possibly for the edges incident with one vertex.

In general, the presence of dangling edges prevents a proof strategy based on the use of a Guthrie colouring of faces. Resolution of the conjecture for some or indeed all $k > 1$ might spring from another proof of the four colour theorem not involving face colours in its proof. In what follows, therefore, we shall keep in mind the general case as well as the instance $k = 0$. For example, the general case may conceivably yield to a proof by induction on k .

The following approach may be found in [Loupekine and Watkins (1)].

The Y Property

Definition. A cubic graph G is said to have the *Y-property* if it has a vertex labelling $X: V(G) \rightarrow C_2$ such that each vertex is adjacent to exactly one other having the same label. See fig 2.6.

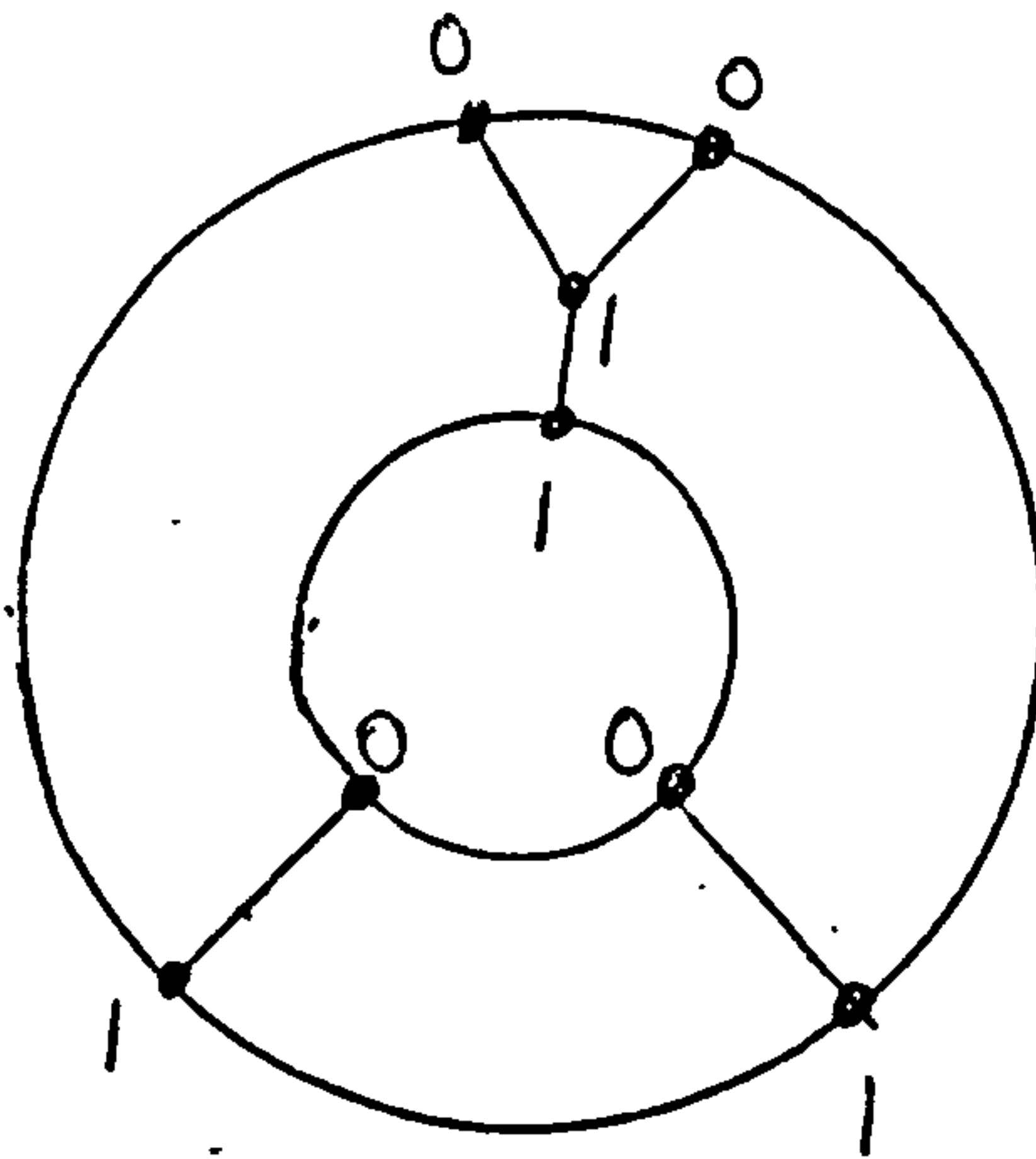


Fig 2.6

Theorem. 2.3

A graph with the Y-property is Tait colourable.

Proof

We shall construct a covering set of disjoint even circuits. Assume that X is a labelling with the required property. Begin at vertex v labelled, say, 0. Then proceed to an adjacent vertex labelled 1. Continue in this way to grow a circuit labelled 0 and 1 alternately. The Y-property ensures that we never cross the path, but since the graph is finite, we must arrive back at vertex v after an even number of steps. (See Fig 2.7).

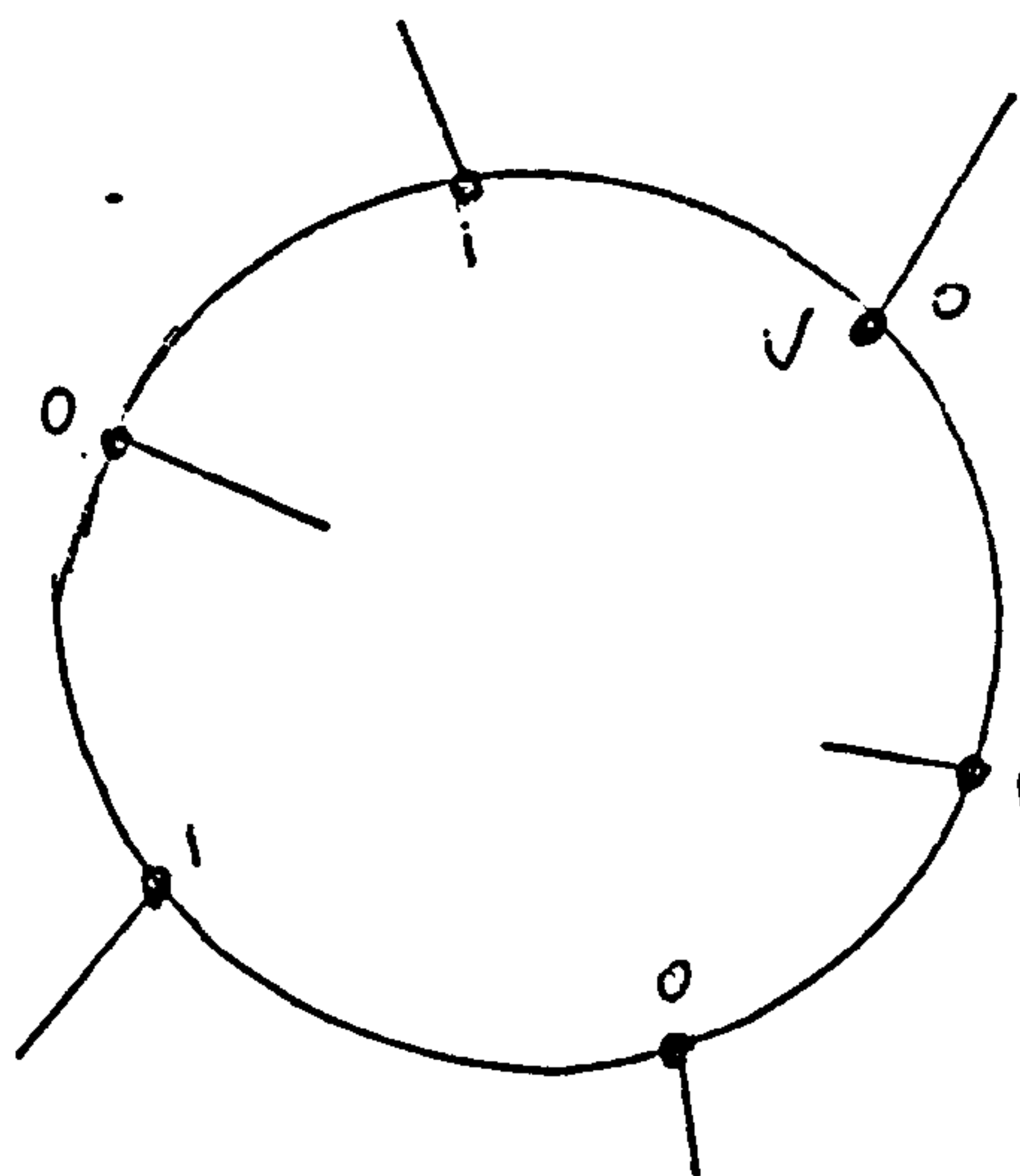


Fig 2.7

Next we choose any vertex not already in such an even circuit, and grow that circuit. Again we generate an even circuit, which must be disjoint from the first, since any vertex on both circuits would be adjacent to three vertices with the opposite label. In this way we proceed to cover the whole vertex set of the given graph. If we assign colours a, b alternately to edges around these even circuits and then colour the other edges with colour c , the colours a, b, c furnish us with the desired Tait colouring. ■

We see from Fig 2.8 that the Y-property is a stronger condition than Tait colourability.

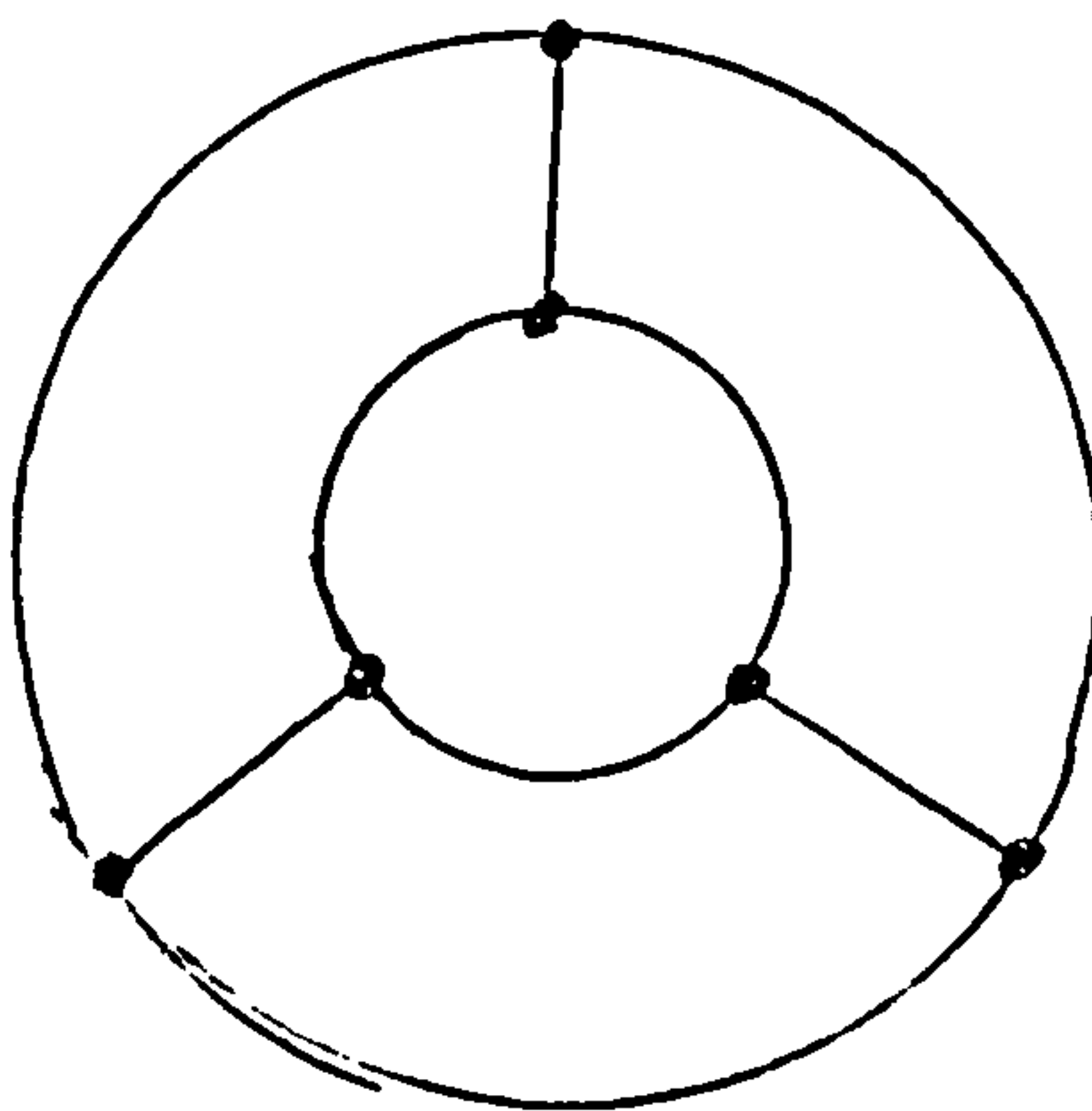


Fig 2.8

The graph in the figure is Tait colourable, but does not have the Y-property. The class of graphs with the Y-property is somewhat sparse, and they bear no obvious relation to any class of graphs with other specific features. But we shall see as a corollary of the next theorem, that all face-even cubic plane graphs have the Y-property. If a plane graph is Tait colourable, then it is always possible to inflate some or all its vertices with triangles so the inflated graph has the Y-property. See Fig 2.9.

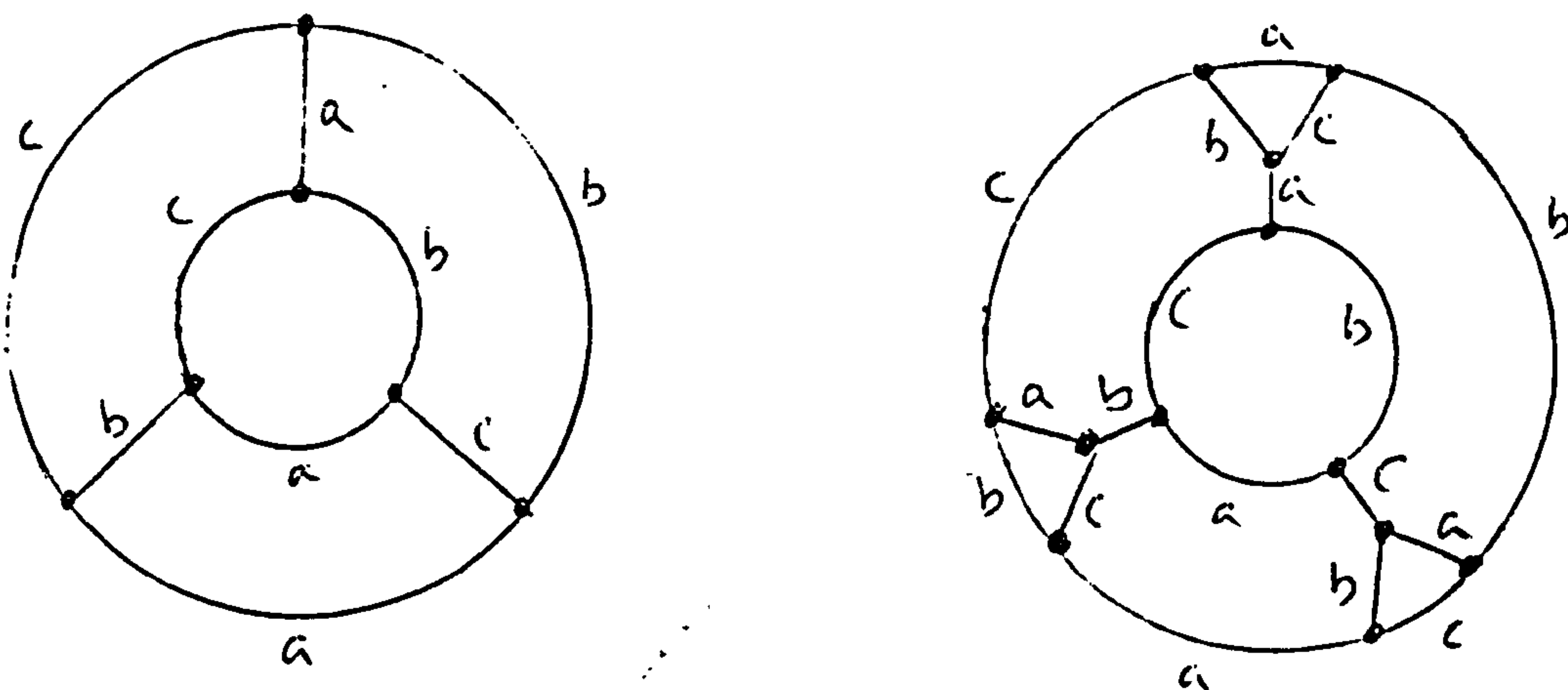


Fig 2.9

The construction is as follows. Given a Tait colouring, we inflate with a triangle those vertices at which the sense of rotation of the colours a, b, c at incident edges is anticlockwise and extend the Tait colouring to a Tait colouring of the new triangles in the obvious way. We see that the sense of the colours of the new triangles joins up with those of the old edges, making a Tait colouring of the inflated graph where the sense of rotation of colours is clockwise round every vertex. Thus the sense is consistently anticlockwise as we go round any face, and so the graph is triple. (See fig. 2.9). We shall see that this property in turn implies that the graph is Watkins colourable (Theorem 2.6), and hence it has the Y-property (Theorem 2.5).

Definition

A spanning set S of disjoint even circuits of a cubic graph is called a *uniform covering* if every circuit in the graph intersects S in an even number of edges.

Theorem 2.4

A cubic graph G has the Y-property if and only if it has a uniform covering.

Proof

Assume that G has the Y-property. As in the proof of Theorem 2.3, construct a spanning set S of even circuits whose vertices are all labelled alternately 0 and 1. Edges not belonging to S have the same label at both ends. Now, along an arbitrary circuit, the number of alterations of vertex label from 0 to 1 or from 1 to 0 must be even, and these changes can occur only along edges in S . Hence, each circuit intersects S in an even number of edges, and S is uniform.

Conversely, assume that there is a uniform covering, S . Take one of the circuits in S , and label its vertices alternately 0 and 1. Then take a vertex v adjacent to this circuit. See Fig 2.10.

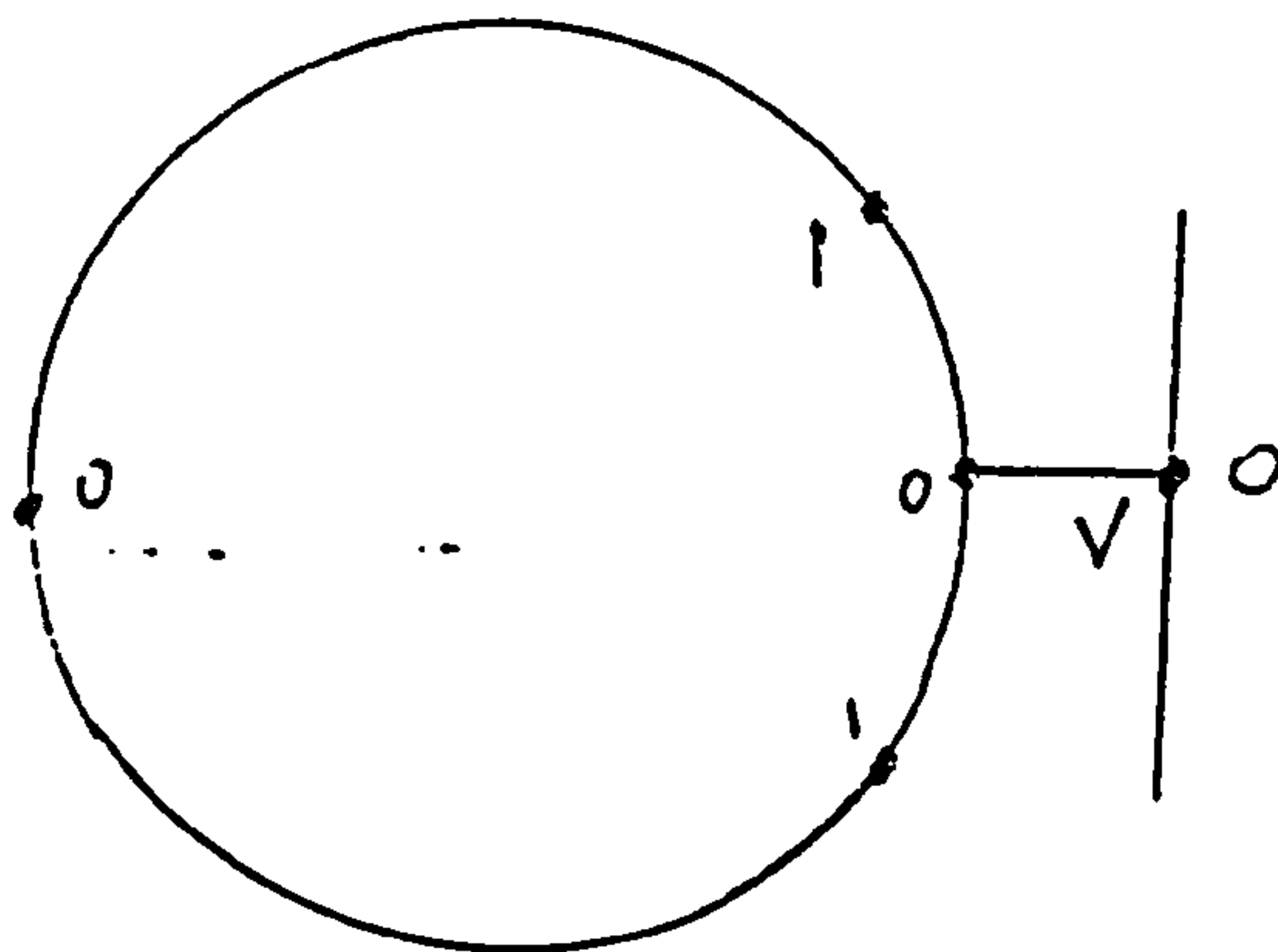


Fig 2.10

Give v the same label as the vertex to which it is adjacent in the circuit already labelled. Then label the vertices alternately 0 and 1 along the circuit in S to which v belongs. Continue in this fashion until all the vertices have been labelled. It is easy to see that the graph has the Y-property if all the edges not in S have the same label

at both ends. Suppose we have just, for the first time, labelled a vertex that is adjacent to a vertex in one of the already labelled circuits of S - with a different label. This gives us a circuit with exactly one interchange between 0 and 1 that does not occur in a circuit of S . This circuit, therefore, intersects S in an odd number of edges, contrary to hypothesis. This completes the proof. ■

Corollary 2.2

If G is a face-even cubic map, then G has the Y-property.

Proof

Let G be such a map. By (Saaty and Kainen, Theorem 2.5), G can be properly face 3-coloured, say with colours a, b, c . (See Fig 2.11.)

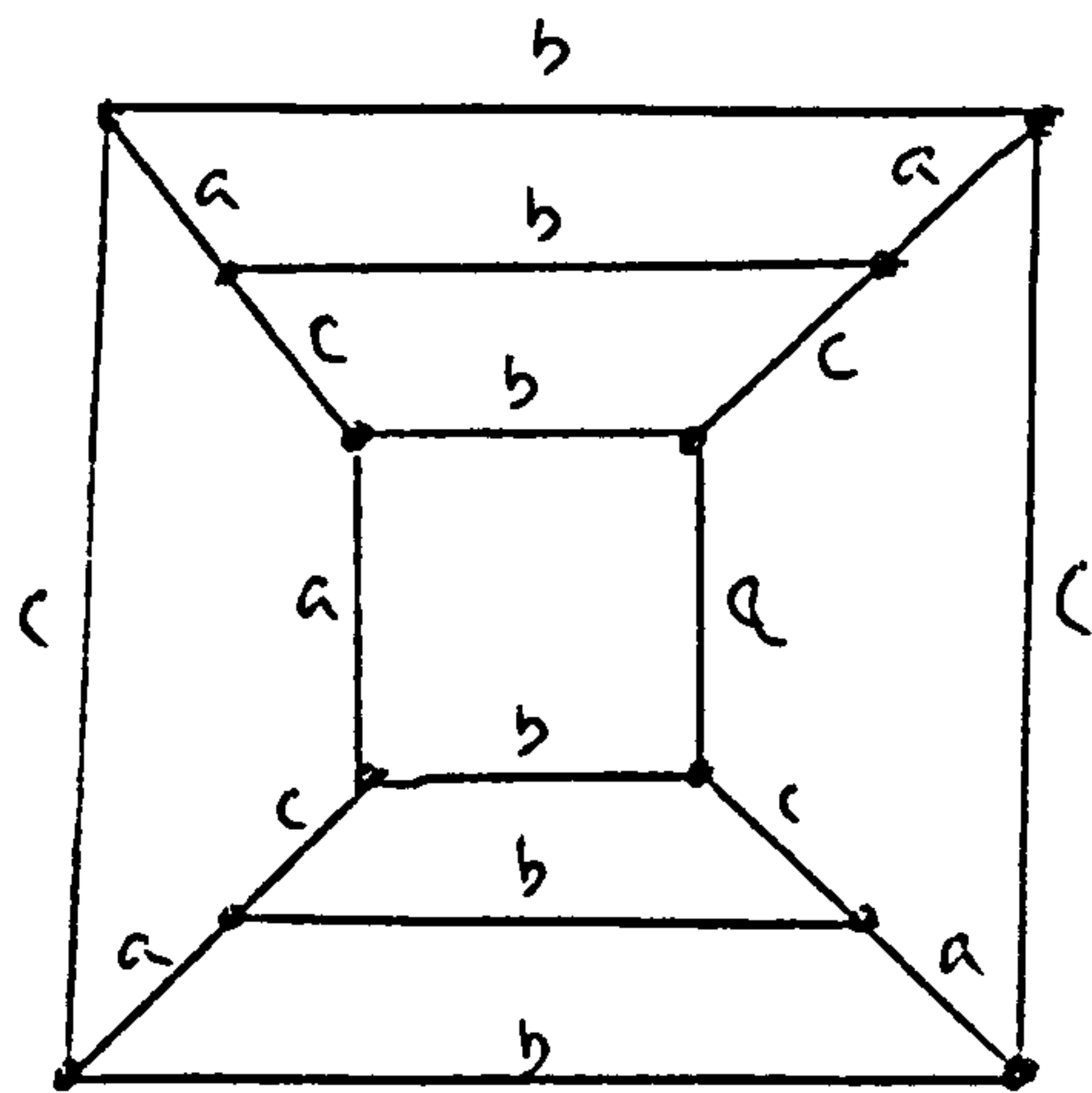


Fig 2.11

Then the boundaries of the faces coloured (say) a constitute a spanning set S of disjoint even circuits.

Now the boundary of any face of G clearly intersects S in an even number of edges, while any circuit in G is the symmetric difference of a (finite) set of face boundaries. Thus, S is a uniform spanning set. ■

Definition

A cubic graph is *Watkins colourable* if one can 4-colour its vertices so that the neighbours of any vertex v have three colours, different from each other and different from the colour of v itself. (See Fig 2.12.)

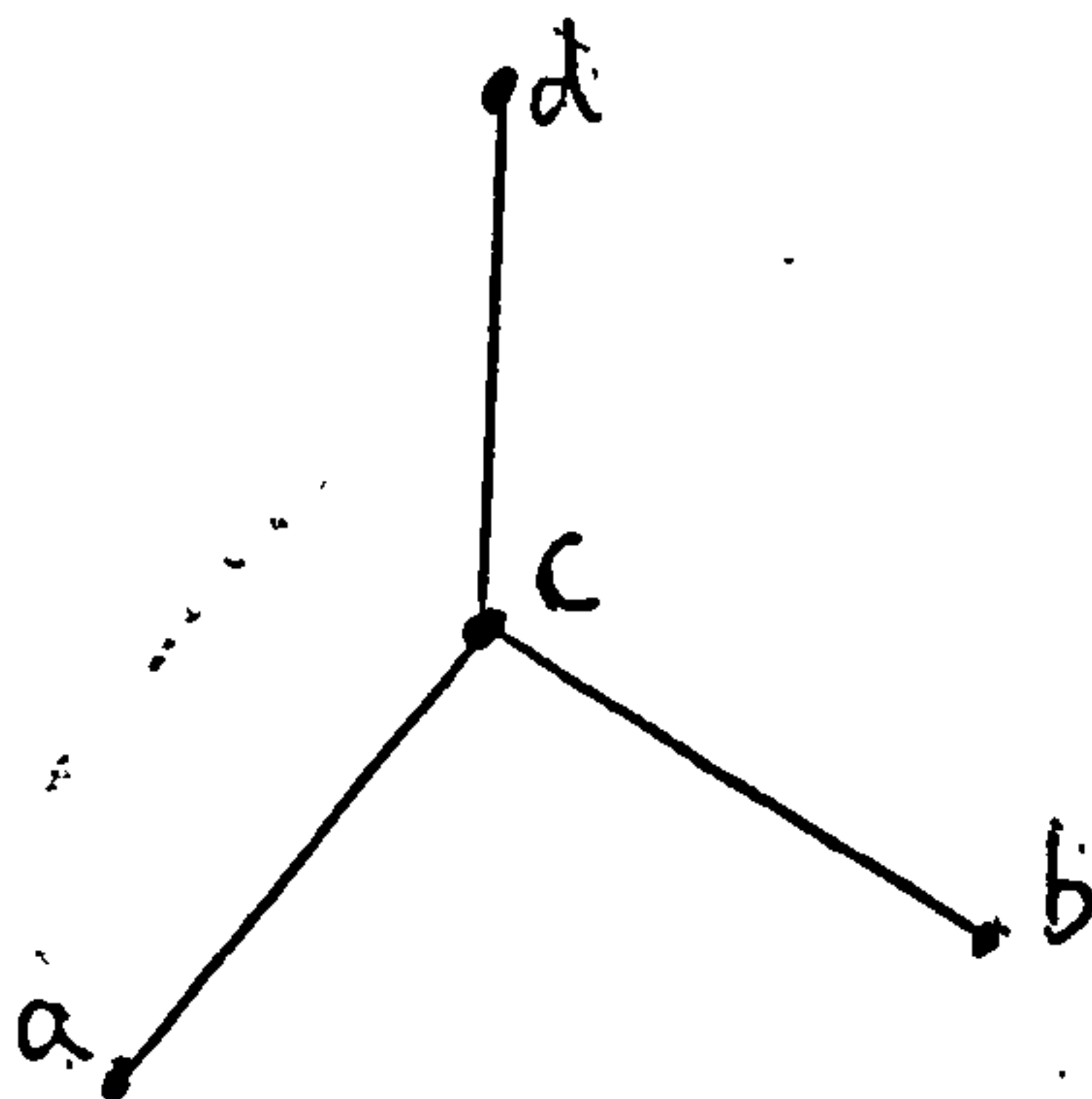


Fig 2.12

Theorem 2.5

A cubic graph that is Watkins colourable has the Y-property.

Proof

Suppose that the graph has been Watkins coloured with colours a, b, c, d . Label the vertices coloured a or b with 0, and those coloured c or d with 1. Then, since each vertex is adjacent to vertices taking the three other colours, it must be adjacent to just one having the same binary label as itself. ■

That the converse of this is false is illustrated by the graph of order 8 in Fig 2.6. That it has the Y-property is demonstrated by the given labelling. We quickly reach an impasse in any attempt to construct a Watkins colouring.

If a bridgeless planar cubic graph is triple, then there is a Tait colouring, the *Ringel colouring*, in which a, b, c occur in clockwise order

round every vertex and consequently in anticlockwise order round every face.

Theorem 2.6

Any triple map is Watkins colourable.

Proof

Let M be a triple map. First, colour the edges of M with the colours a, b, c , using a Ringel colouring, T .

We now generate a Watkins labelling, as follows. Regard a, b, c as the non-identity elements of the Klein group K , and let w be an arbitrary vertex of M . For each vertex v of M and for each edge sequence

$$P = (e_1, \dots, e_p) = (\{w, v_1\}, \{v_1, v_2\}, \dots, \{v_{p-1}, v_p\})$$

where $v_p = v$, let $X(v, P) = \sum_1^p T(e_i)$ (thus, if \emptyset is the empty path, then $X(w, \emptyset) = d$). The fact that T is a Ringel colouring implies that the sum of the edge labels round any face (and hence round any circuit) is the identity in K , so that X depends only on v ; thus, X defines a function X' from $V(M)$ to K . Since the three edges of M incident with any vertex are labelled with the three non-identity elements of K , it follows that X' is a Watkins colouring. ■

The converse is false, as the map of Fig 2.13 shows.

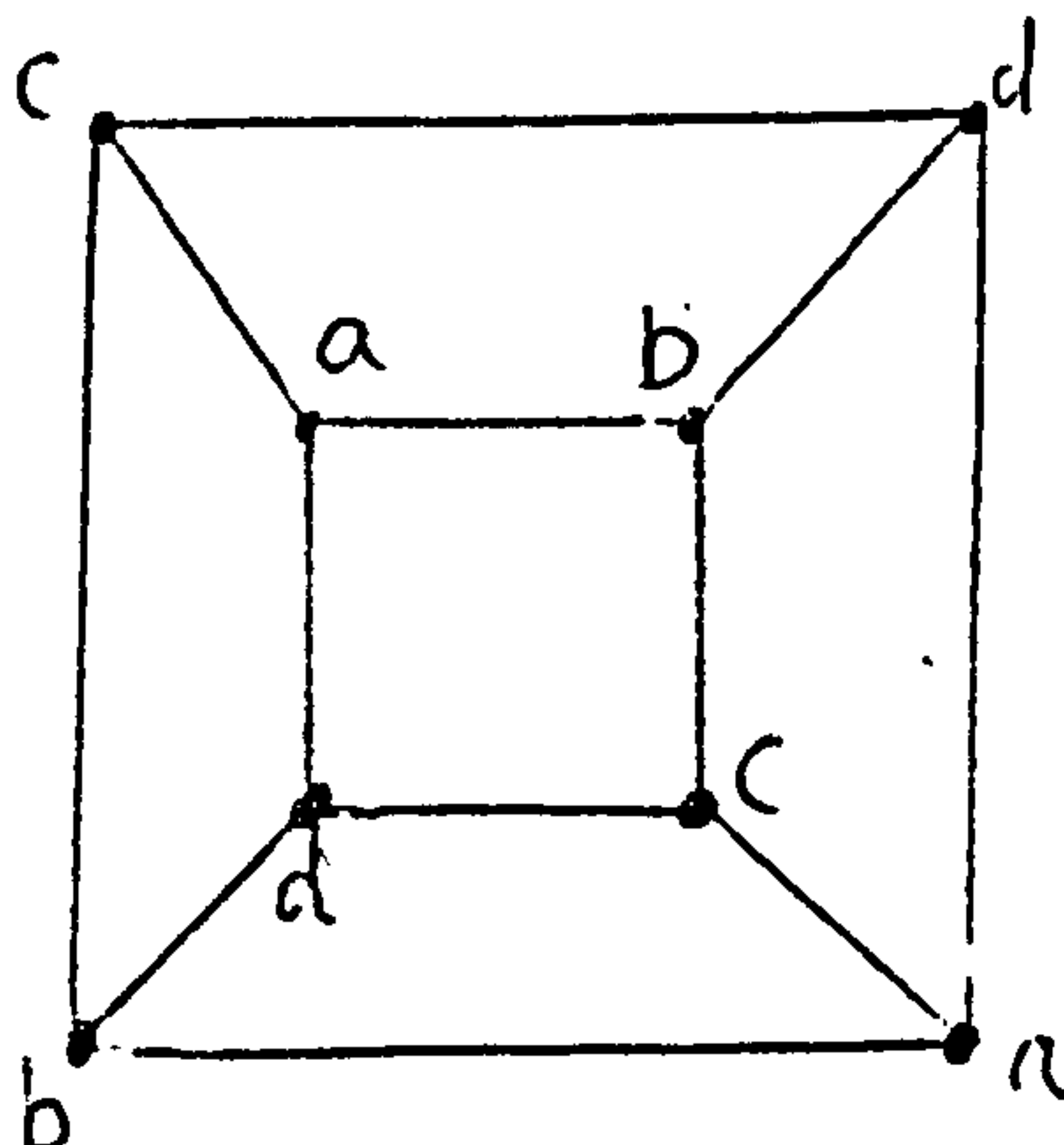


Fig 2.13

The Nesting of Cubic Graphs According to Labellings

The existence of a Tait colouring of a cubic graph is equivalent to the existence of a 1-factorisation, and therefore implies the weaker property that there exists a 1-factor. This in turn implies the still weaker condition that the edges can be coloured with two colours so that each colour appears at each vertex.

The results of this chapter so far can be summarised by saying that classes of cubic graphs can be nested according to their possible labellings. Let us index these labelling properties (1)...(7) as follows:

- (1) The edges can be coloured with two colours so each colour appears at each vertex.
- (2) There is a 1-factor.
- (3) There is a Tait colouring.
- (4) The Y-property.
- (5) The uniform circuit property.
- (6) Watkins colourability.
- (7) (for plane graphs) The triple property.

The conclusions can be summed up by the following implications:

$$(7) \Rightarrow (6) \Rightarrow (5) = (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).$$

Inflating a vertex to a triangle does not affect properties (1), (2) or (3). Given that a graph can be Tait coloured, say with colours a, b, c, we inflate those vertices (to a triangle) where a, b, c, appear in anticlockwise order. Then we extend the former Tait colouring to the new triangle so that a, b, c appear in clockwise order everywhere. Thus the

graph is now triple, and since $\langle 7 \rangle \Rightarrow \langle 6 \rangle \Rightarrow \langle 5 \rangle = \langle 4 \rangle$, these properties hold too.

Angle Labellings and the Klein Group

With each vertex of a cubic we associate six angles; three we call *interior*, the rest *exterior*. (See Fig 2.14, where a_1, a_2, a_3 are the interior angles, and b_1, b_2, b_3 the exterior angles, associated with the vertex v .)

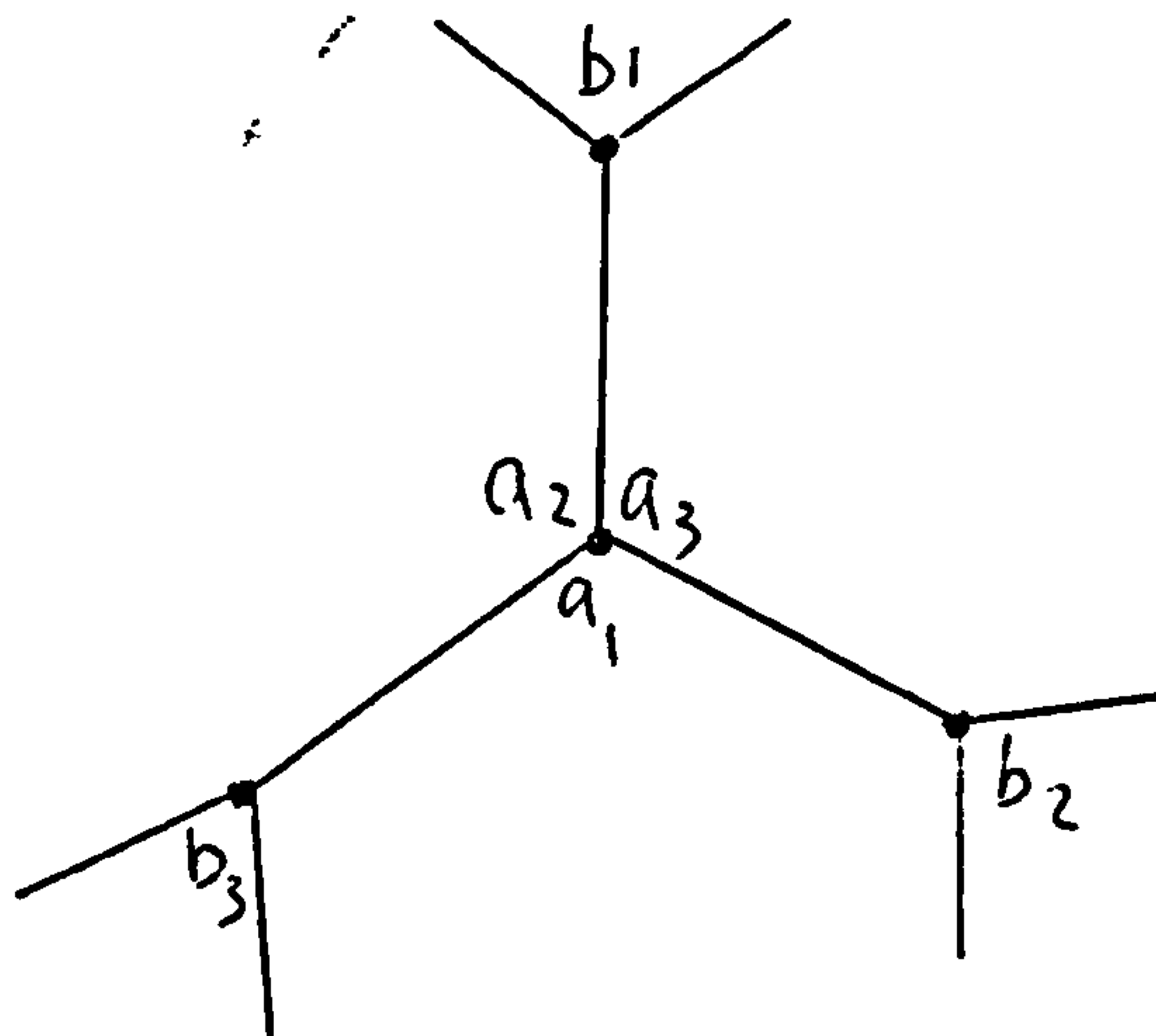


Fig 2.14

Theorem 2.7

A plane cubic graph G is Tait colourable if and only if there is an angle labelling $X: A(G) \rightarrow C_2$ such that at each vertex the six associated labels satisfy the congruence:

$$\sum_1 X(a_i) + \sum_1 X(b_i) \equiv 2 \pmod{4}.$$

Proof

Since the labels are restricted to 0 and 1, this means that the sum is either +2 or -2.

Let G be a graph that is Tait colourable. By Tait's Theorem, there is a covering set S of disjoint even circuits. We define a function $\gamma: V(G) \rightarrow \{0, 1\}$, which assigns values 0, 1 alternately to vertices

along each circuit of S . Next, we let Z be any function from $E(G) - S$ to C_2 . Finally, we define a labelling of the angles of G as follows. Let v be a vertex and e be the edge in $E(G) - S$ that is incident with v ; then the two angles at v that are separated by e are labelled with $Y(v)$, while the other angle at v is labelled with $Z(e)$. (See Fig 2.15.)

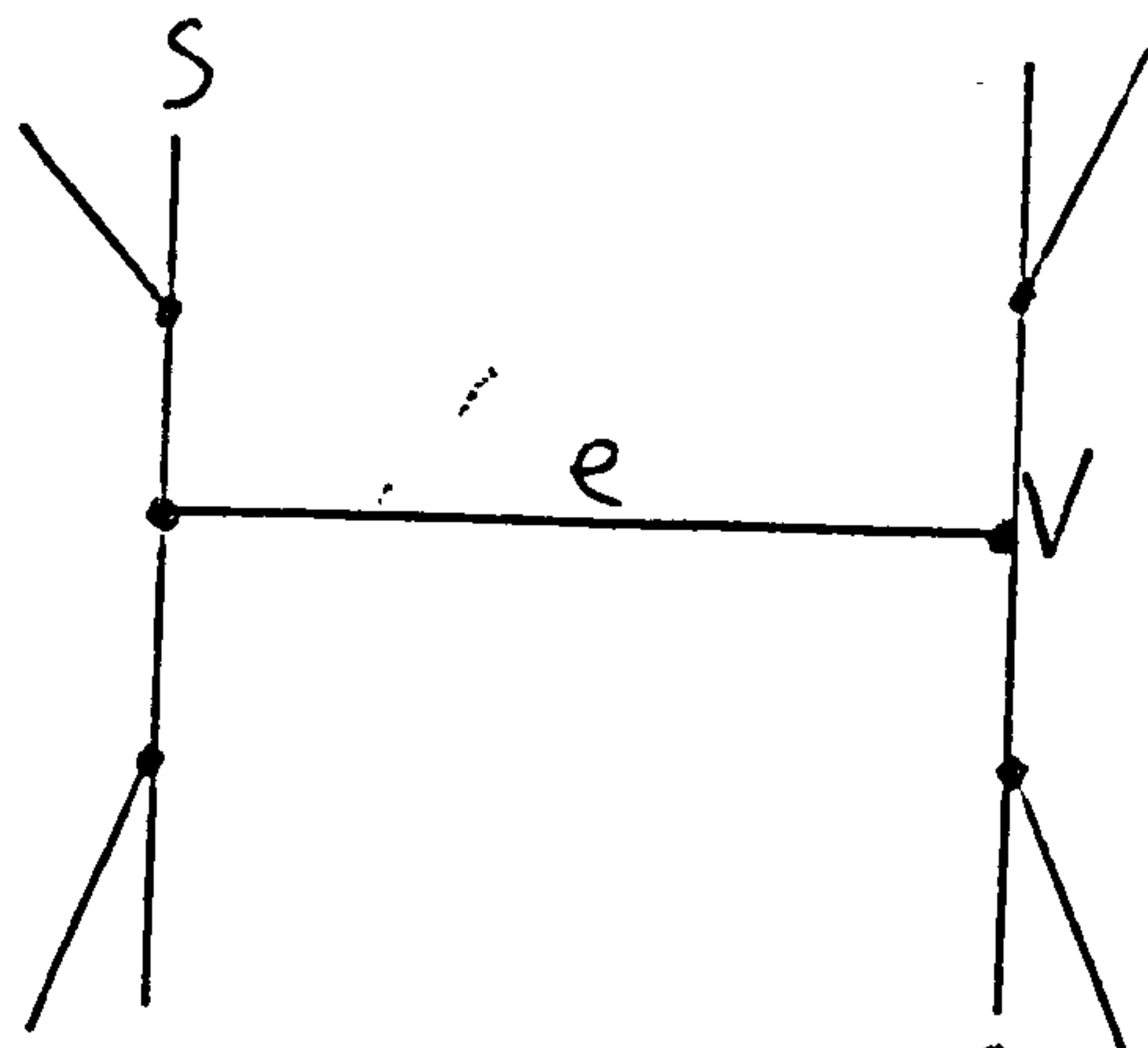


Fig 2.15

Therefore, the three internal labels at v are $Y(v)$ (twice) and $Z(e)$, while the three external labels are $1 - Y(v)$ (twice) and $Z(e)$; thus, the required congruence holds at v .

To prove the converse, assume that G is a plane cubic graph with an angle labelling satisfying the system of congruences. We inflate each vertex to a triangle to get a new graph G^* , and use the angle labelling of G to label the vertices of G^* as shown in Fig 2.16.

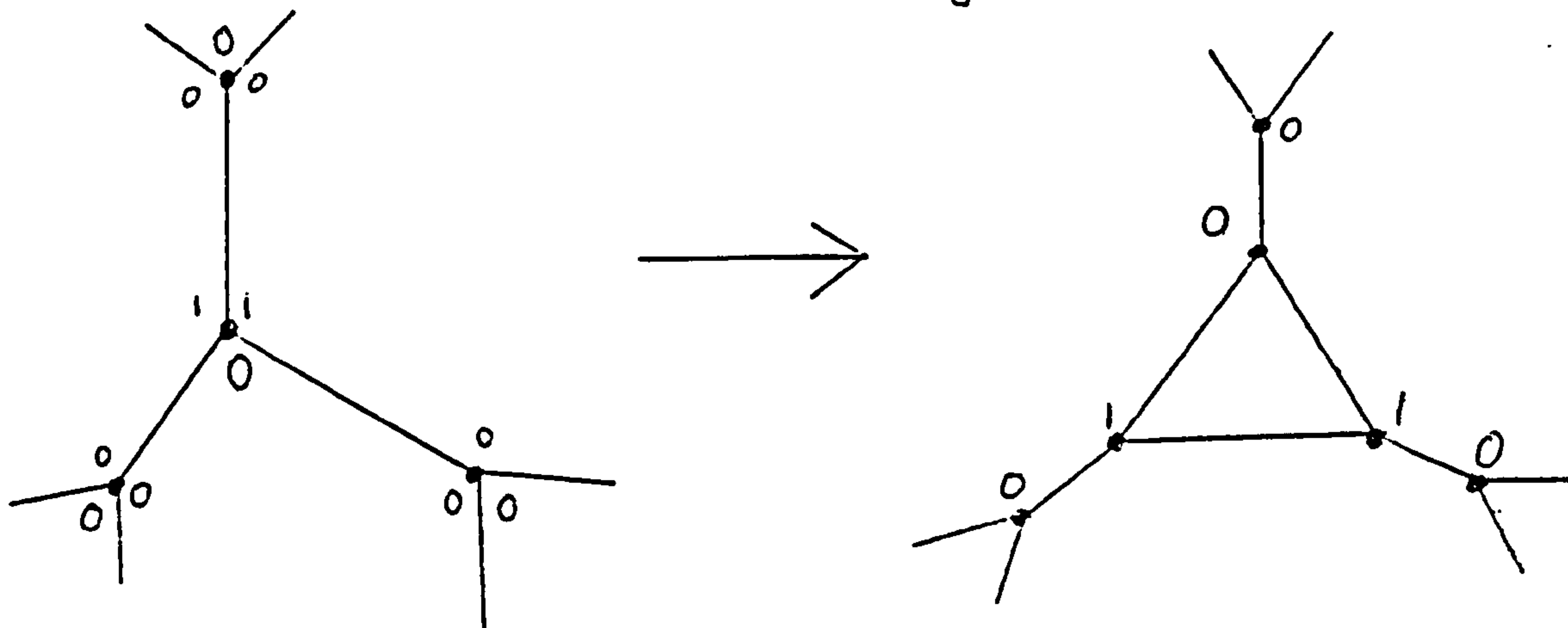


Fig 2.16

Since inflation of vertices to triangles does not affect Tait colourability, G is Tait colourable if and only if G^* is. The original congruences that hold at each vertex of G for the angles now hold at each triangle of G^* for the vertices. For a given triangle in G^* there are essentially four possible labellings of the three vertices using 0 and 1. However, since we can interchange 0 and 1, we discuss only the two triangles as labelled in Fig 2.17.

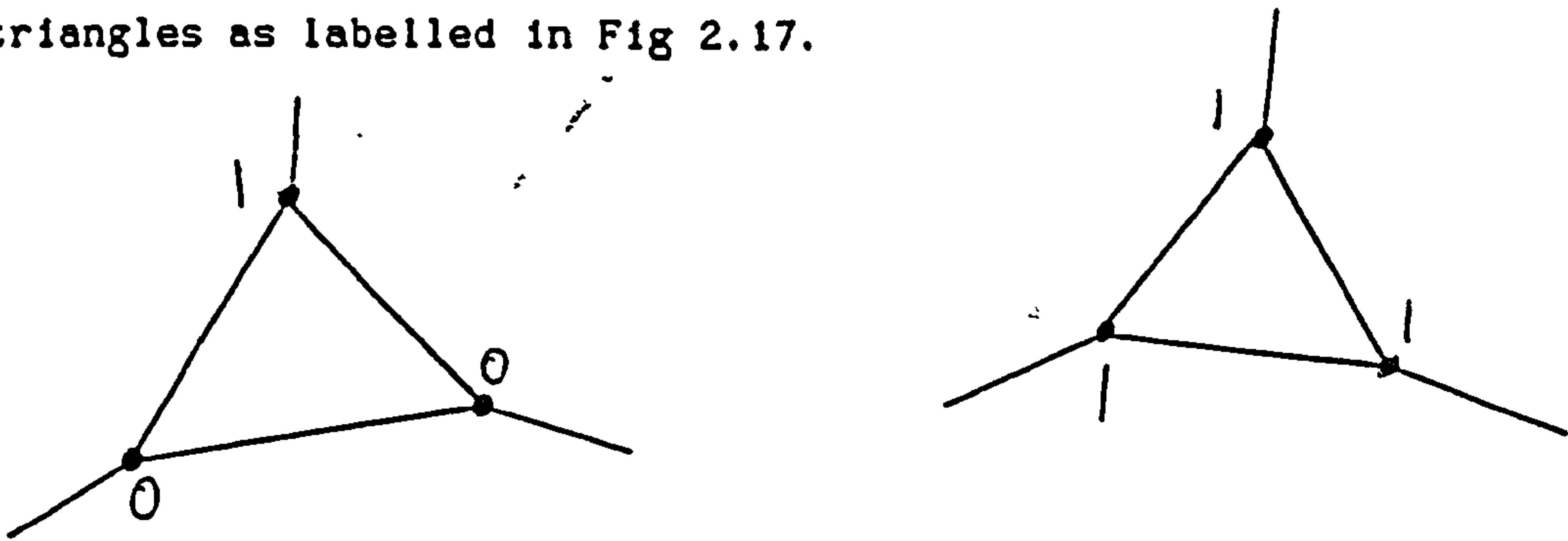


Fig 2.17

For the first triangle in fig.2.17 to satisfy the congruence, the three vertices adjacent to it must all be labelled 1. (So the sum is -2). This means that each vertex in the triangle has just one neighbour having the same label. For the second triangle to satisfy the congruence, exactly two of the vertices adjacent to it must be labelled 0. In this case we contract the triangle back to a single vertex with label 1. (See

Fig 2.18.)

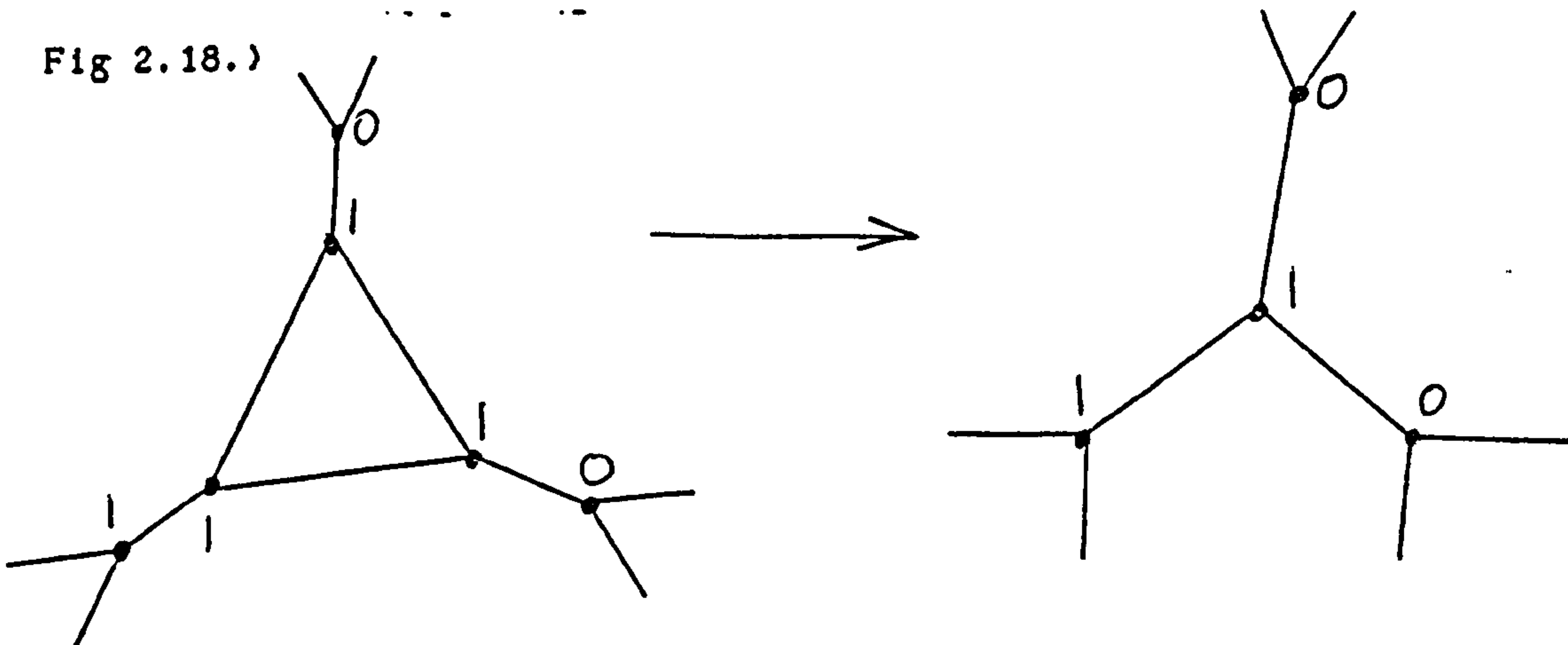


Fig 2.18

This vertex also has one neighbour that shares its label and two neighbours with the opposite label.

Thus we now have a graph G^{**} which has the Y-property. By Theorem 2.3 it is Tait colourable, and since G^{**} can be derived from G by inflating some vertices to triangles, it follows that G is Tait colourable. ■

CHAPTER THREE

Snarks

According to [Vizing] all graphs of maximum degree R can be edge coloured with at at most $R + 1$ colours. We obviously need at least R colours, to cope with the vertices of that degree. Therefore, after [Fiorini and Wilson], we can split all graphs of maximum degree R into class one and class two according as the chromatic index is R or $R+1$, respectively.

Definition [J.J.Watkins, R.J.Wilson]

A snark is a connected bridgeless cubic graph with chromatic index 4 (i.e., of class 2).

The name "snark" was first used in this connection by [Gardner]. As well as being related, as we shall see, to the four colour theorem (via Tait's conjecture), the study of snarks is now a busy area of research as a subject in its own right.

Theorem 3.1

Every face-even p.c.b. map is of class one.

Proof

This follows at once from Theorem 2.3 and Corollary 2.2 (to Theorem 2.4). ■

We note [Celmins and Swart] that the conditions "connected and bridgeless" can be replaced by the exactly equivalent single condition "2-connected".

The attitude as to what snarks are considered trivial and what non trivial depends on the extent that snark study has reached. We suggest that the following definition, at least, may stand the test of time:

Definition

A snark is *prime* if it has no proper subgraph which is also a snark.

If a snark is not prime, we say it is *imprime*.

A graph with a bridge is worse than imprime. It is *trivial*.

Thus a snark which can be split by a small cutset into a pair of graphs, each containing at least one circuit and at least one of which is a snark, is imprime. It is an easy matter to show that 3-separating edges make a snark imprime and recently, trivial 4- and 5-cutsets were shown by [Cameron, Chetwynd and Watkins] to make a snark imprime.

[Tait] proved that the proposition that all p.c.b. graphs can be edge 3-coloured is equivalent to the four colour theorem. He has given his name to edge 3-colourings. What Tait may not have realised was that there are counterexamples such as the Petersen graph (Fig 3.1) and (Fig 3.2) if we relax the conditions "planar" and "bridgeless".

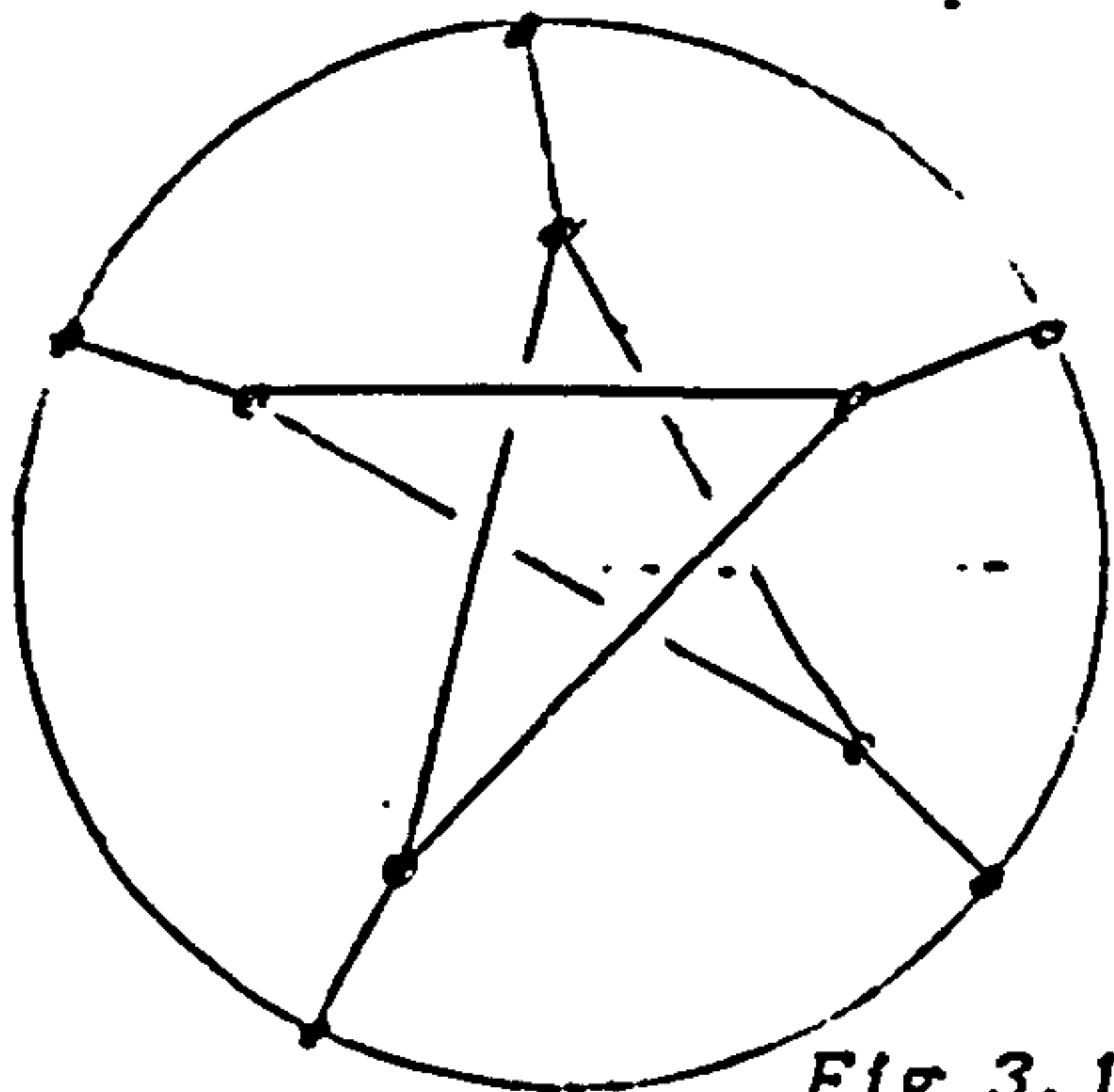


Fig 3.1

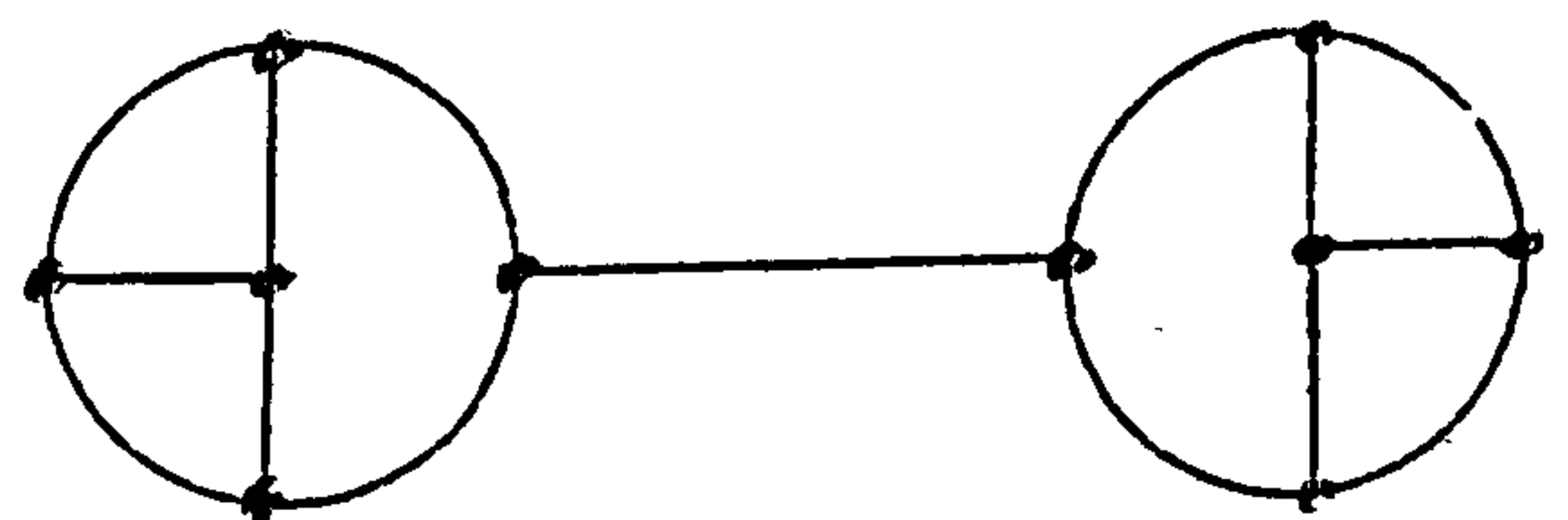


Fig 3.2

We shall refer to the above result as *Tait's Theorem*. It is proved by showing that, to any Guthrie colouring of a cubic map, there corresponds a Tait colouring and *vice versa*. Indeed, if the Guthrie colouring uses the four elements of the group K , then any edge of the map may be assigned the non-identity element which is the sum of the face colours that it

separates. We shall refer to this as the *corresponding* Tait colouring. Conversely, given a Tait colouring, there are exactly four Guthrie colourings to which it corresponds, determined by the choice of colour allocated to some arbitrary face.

Theorem 3.2

Any graph having just one vertex of degree 1 or 2, the others being of degree 3, is of class 2.

Proof

This is just the case $R = 3$ of Theorem 2.2. ■

The next theorem is due to the author and J.J.Watkins; its proof uses the connection between Guthrie colourings and the corresponding Tait colourings.

Theorem 3.3

Let G be a cubic bridgeless graph which is plane except for one pair of crossing edges. Then the four colour theorem implies that G is of class 1.

Proof

Suppose we are given a cubic bridgeless graph with crossing point I ; see Fig 3.3(a). Then if we cut both crossing edges at I , we form a hypograph G' with four dangling edges u, v, w, x ; see Fig 3.3(b). We now insert a small square H at the dangling ends, to obtain a cubic bridgeless graph G'' which is also plane, as in Fig 3.3(c).

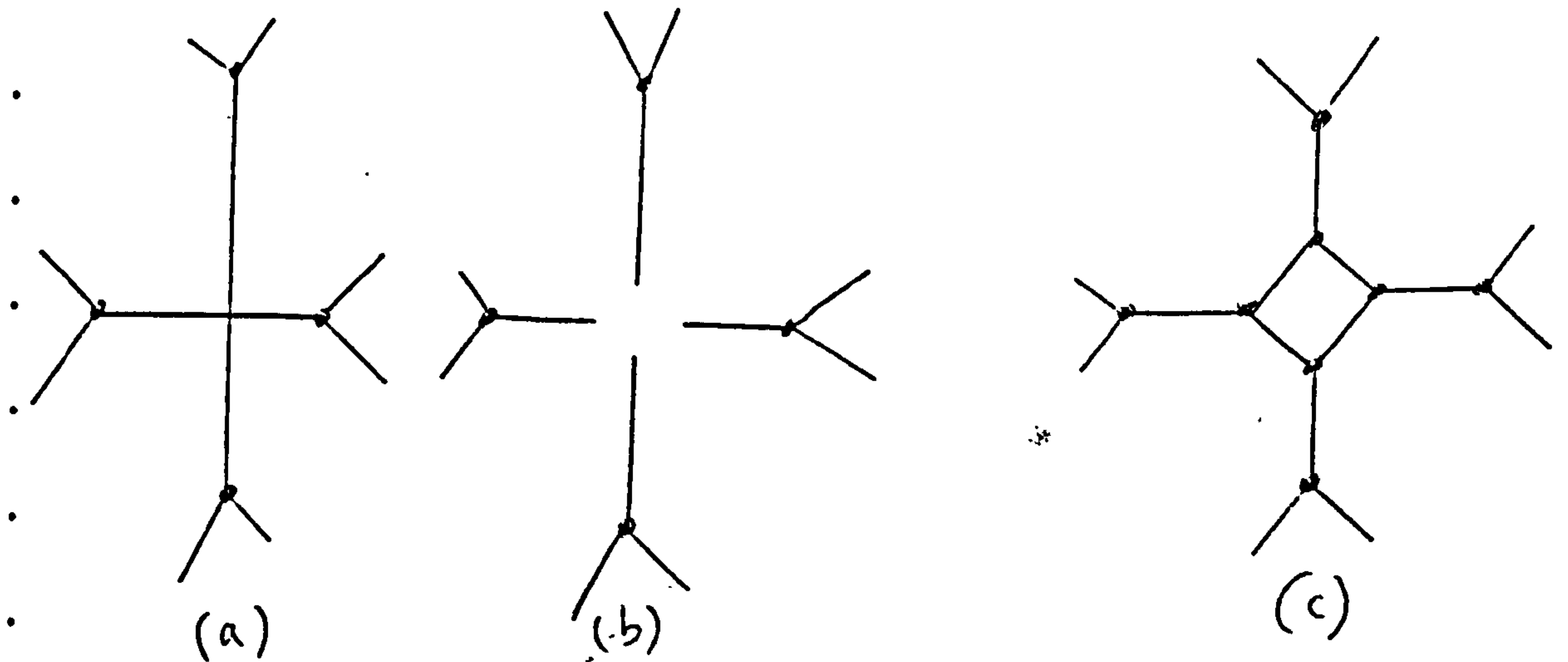


Fig 3.3

Now G'' is Guthrie colourable by the four colour theorem, and so by Tait's Theorem has a Tait colouring with the elements a, b, c of K . Then, up to a permutation of the colours a, b, c , the Parity Lemma implies that the sequence of colours of the edges u, v, w, x is one of (a, a, a, a) , (a, b, a, b) , (a, a, b, b) or (a, b, b, a) .

The first two cases, (a, a, a, a) and (a, b, a, b) , lead directly to a Tait colouring of G , so we only need to deal with the last two cases.

Since they are rotations of each other, we assume without loss of generality that we have case (a, a, b, b) . We remove the edges of H , leaving the hypograph G' , and proceed to interchange the colours of edges in the (a, b) Kempe chain containing edge u - that is, the path consisting of edges alternately coloured a, b . We find, to avoid cases (a, a, a, a) and (a, b, a, b) above, that w must change colour too. So edges u and w must be connected by the Kempe chain. But v and x must be similarly connected, giving a contradiction. Thus one of the Kempe chains does not occur, and we may re-colour G' in such a way that the dangling edges can be reconnected to form an edge-colouring of G . ■

No corresponding result for the case of two crossing points is possible, as Fig 3.4 shows, nor for the case of one crossing point involving three edges (Fig 3.5), since Figs 3.4 and 3.5 are both representations of Petersen's graph. (fig 3.1)

For $n = 2, 3, 4, \dots$ we conjecture that there is no snark everywhere planar except for one edge crossing n other edges (Fig 3.6).

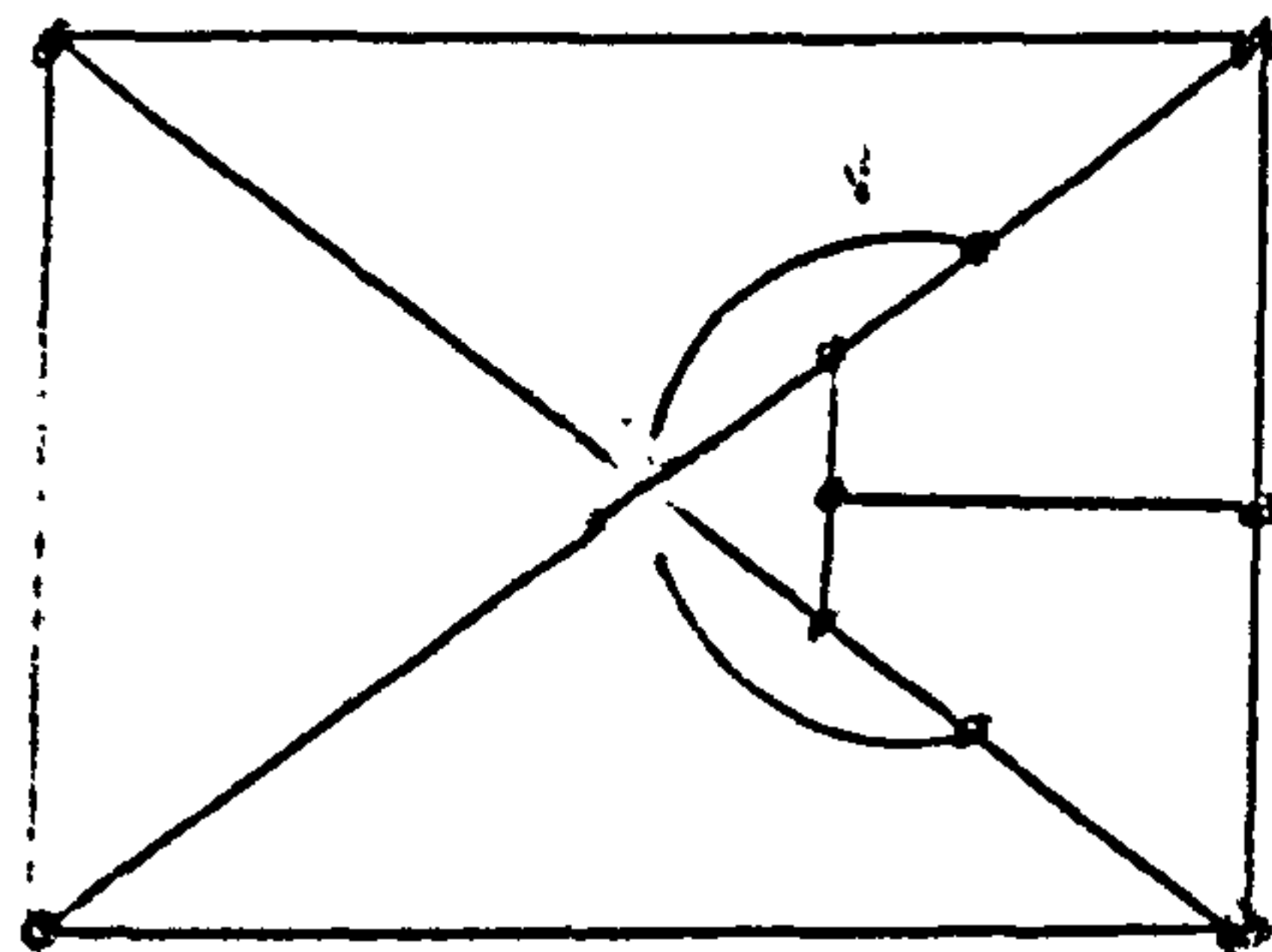
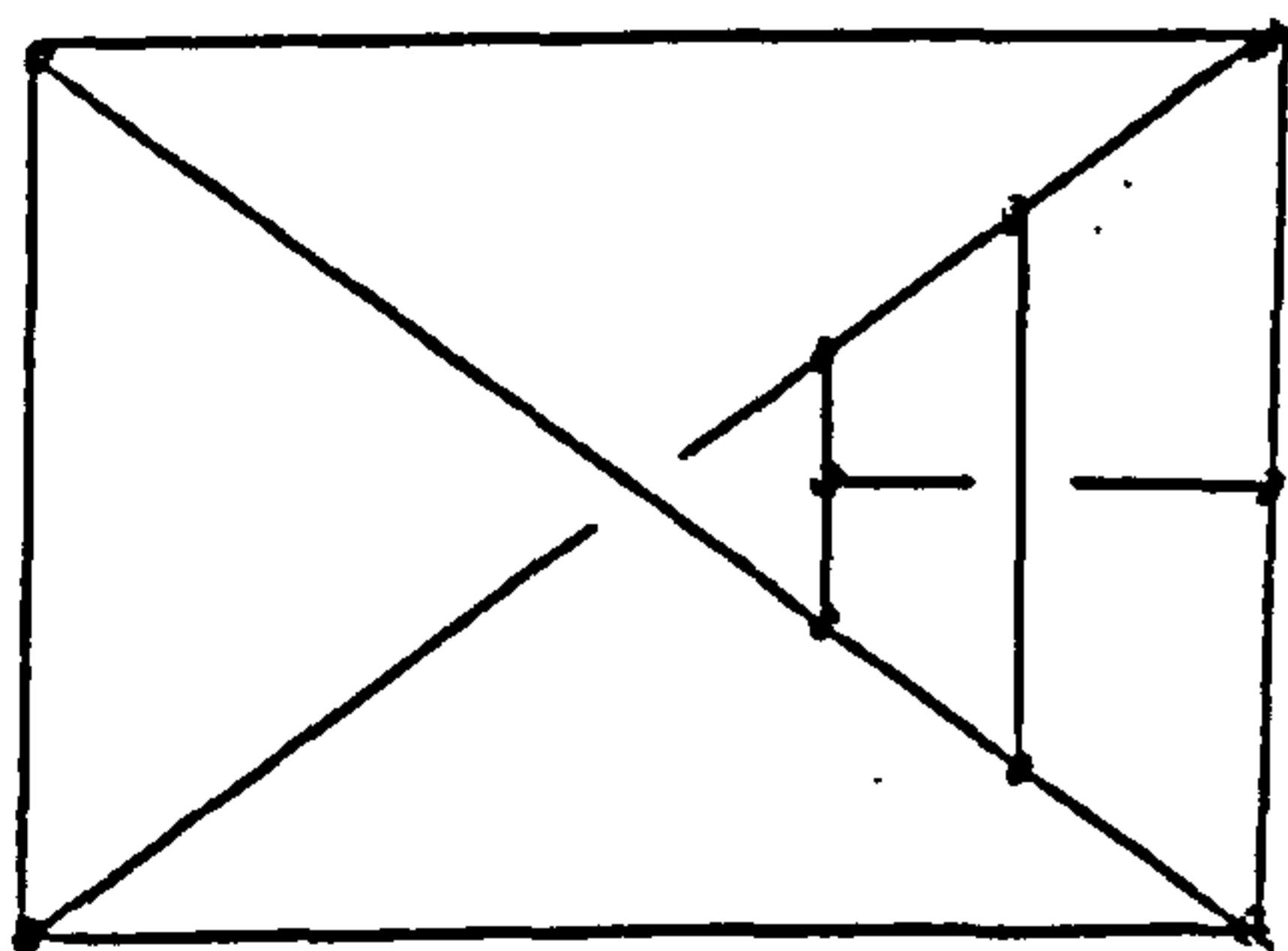


Fig 3.4

Fig 3.5

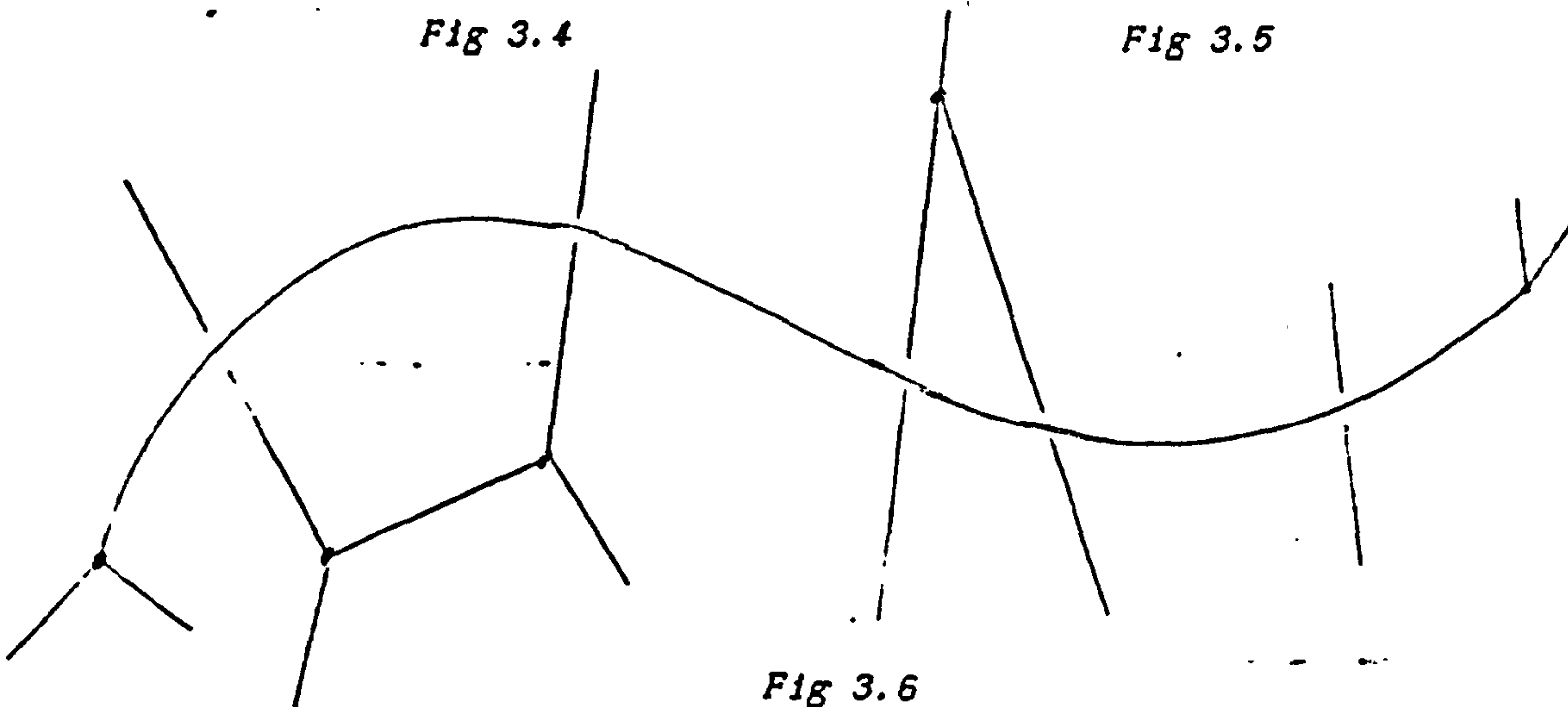


Fig 3.6

Definition

A *region* in a graph drawn in the plane with a finite number of crossing points is a connected component of the complement of the drawing. Thus the set of regions of such a graph will in general depend on the drawing.

Definition

Suppose a non-planar graph G is drawn with crossings, each crossing involving just two edges. Then G is *region colourable* if we can colour the regions of G with four colours so that:

- (a) regions having common line boundaries (i.e., part or whole edges of G) have different colours;
- (b) at each crossing point, either all four colours are present or just two are present, diagonally opposite regions taking the same colour.

(See Fig 3.7.)

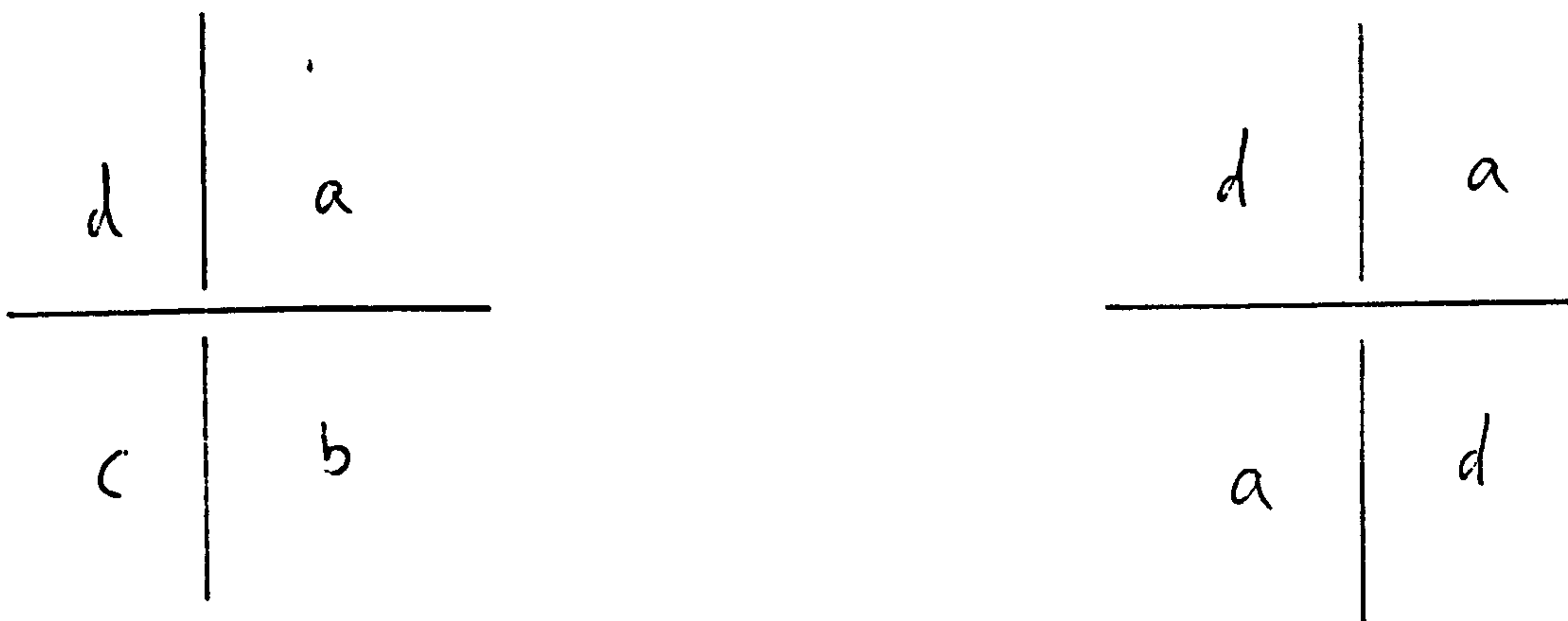


Fig 3.7

The next result shows that region colourings correspond to Tait colourings in much the same way as do Guthrie colourings of plane cubic graphs.

Theorem 3.3

A drawing of a cubic non-planar graph G is region colourable if and only if G has a Tait colouring.

Proof

If the regions are coloured a, b, c, d in K , then the graph can be Tait coloured, each edge or part of edge receiving the colour which is the sum of colours of the two regions which it separates. If the region colours at a crossing point alternate, then clearly the part edges corresponding to the same edge receive the same colour (indeed, both crossing edges take the same colour). If the region colours are a, b, c, d , in some order, the same result follows by observing that the sum of any two of the colours a, b, c, d is equal to the sum of the other two.

Conversely, if we have a Tait colouring then this leads to four distinct region colourings in exactly the same way as in Tait's Theorem. That is, having assigned one region a colour arbitrarily, we perform the inverse operation to that described in the first part of this proof. The proof is analogous to that of Tait's Theorem, with the addition that at crossing points we note that condition (b) of the definition of region colouring is obeyed.

Corollary 3.1

If a cubic bridgeless graph is drawn in the plane in two different ways, each with crossing points, then the drawings are either both, or neither, region colourable.

Proof

Region colourability is equivalent to Tait colourability, which is not affected by the way a graph is drawn in the plane. ■

Corollary 3.2

Let P be the external face of a cubic bridgeless graph G drawn in the plane with points of intersection which do not appear in the boundary of P . Then G has a Tait colouring if and only if it is possible to give the regions

internal to P a region colouring in which at most three colours appear on the boundary of P . Thus, G is of class 2 if and only if the regions bounding P need 4 colours.

Proof

If G is of class 1, then it has a Tait colouring which includes all the edges of the boundary of P . So G has a region colouring in which P is coloured. As points of intersection do not appear in the boundary of P , a region colouring of the regions internal to P extends to a region colouring including P if and only if a colour, distinct from those on the boundary of P , is available for P .

Conversely, suppose P and its boundary regions can be 4-coloured with a region colouring covering the whole of G . Then we can define a Tait colouring of the intersecting edges. ■

Construction of Snark Families

Much has been attributed to the author concerning the construction of snarks [J.J.Watkins and R.J.Wilson]. But the only real contribution is the pair of sets of constructions, the odd and even cases, of the families of Figs. 3.10 and 3.11.

This is well documented in Rufus Isaacs' paper which is included as an appendix to the present work. Isaacs has presented a completely different approach to this bi-family of snarks. Also he presents certain variations which extend the bi-family in a new and interesting way..

But now we consider the technique by which these snarks were first discovered.

Definition

A *pentagram* is a drawing of a 5-circuit in such a way that each edge crosses both the edges to which it is not adjacent. If a pentagram P is part of a drawing in the plane of a cubic graph, then its *surrounding regions* are the regions each of whose boundaries contain two vertices of P . (See Fig 3.8, in which Petersen's graph is drawn in a manner that involves a pentagram, whose surrounding regions are labelled A to E.)

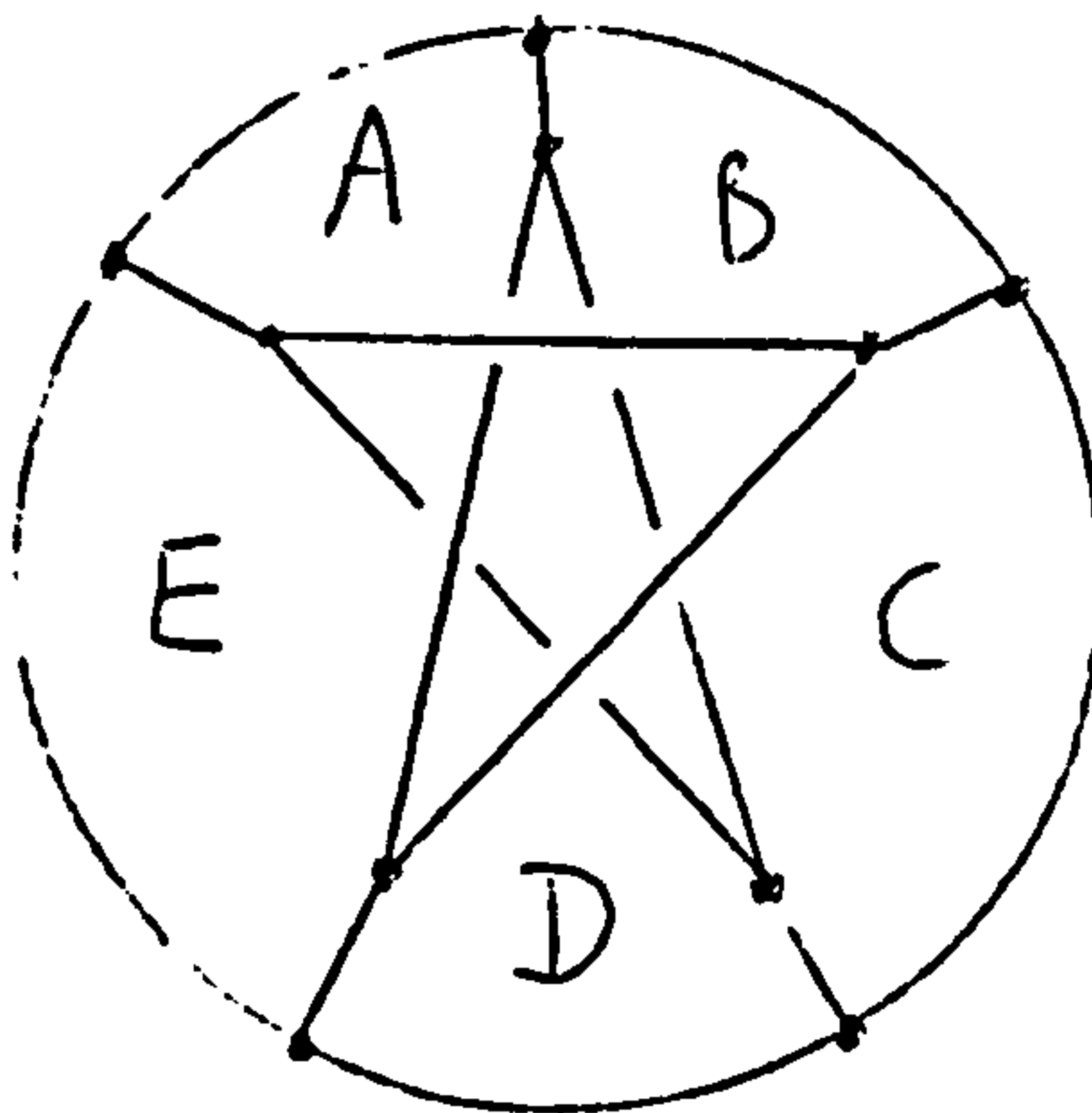


Fig 3.8

Theorem 3.4

Let P be a pentagram in some drawing of a cubic graph G . Then in any region colouring of G , all four colours appear in the surrounding regions of P .

Proof

Petersen's graph is well-known to be a snark. Now if the regions within a pentagram could be coloured in such a way that the surrounding regions could have just three colours, then this would be true of the pentagram in Fig 3.1, contradicting Corollary 3.2. ■

This theorem allows us to construct an infinite family of snarks, using an odd number of pentagrams arranged in a cycle. Consider, for example, the following arrangement of three pentagrams (Fig 3.9).

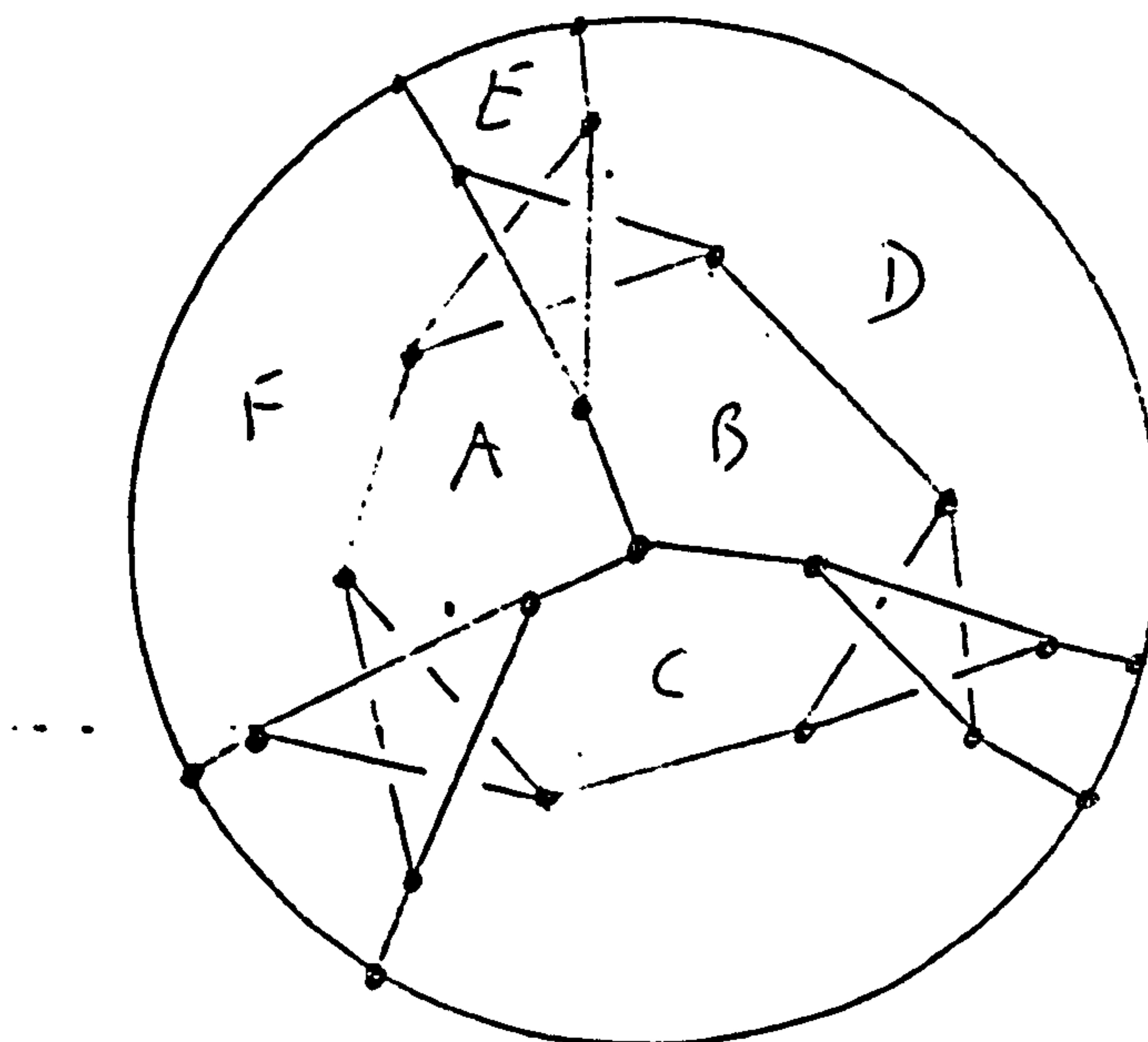


Fig 3.9

Suppose this drawing had a region colouring. By Theorem 3.4, one of the regions A,B must be coloured with the colour of the external face, as none of the regions D,E,F are. But the same argument applies to the pairs B,C

and C,A. This is plainly impossible. So Fig 3.9 has no region colouring, and by Theorem 3.3 it is therefore also a snark.

More generally, consider any drawing of a cubic graph, in which $2k+1$ pentagrams are arranged in a cycle, their $4k+2$ edges external to the cycle being incident with a circuit and their $4k+1$ edges internal to the cycle being linked via other edges and vertices drawn without further crossings (see Fig 3.10). If such a drawing had a region colouring, Theorem 3.4 would force the regions Z_i ($i = 1, \dots, 2k+1$) to take the colour of the external face alternately, contradicting the fact that there are an odd number of them. Thus, any cubic graph that can be drawn in this way is a snark.

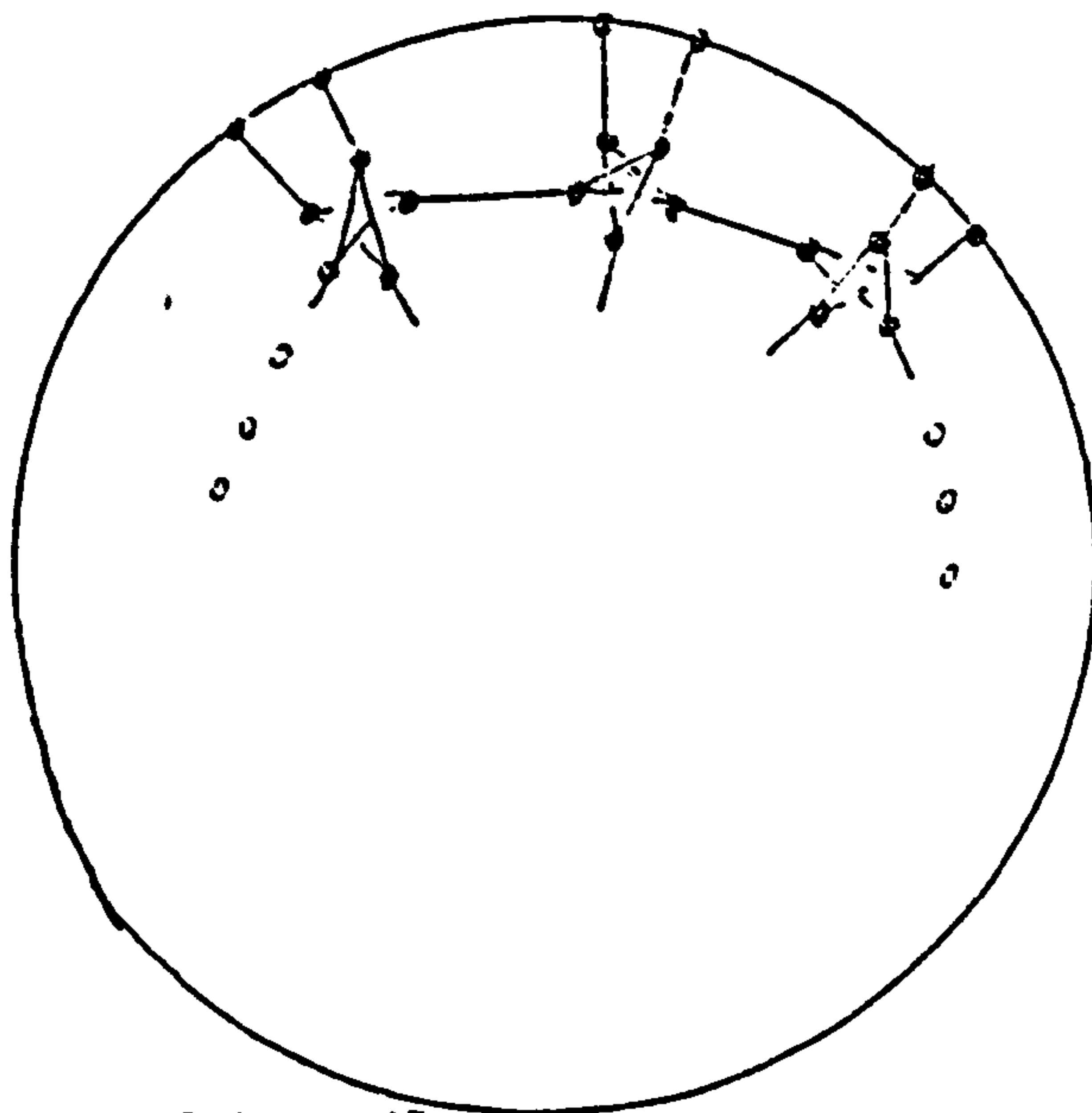


Fig 3.10

In the even case [R.Isaacs] we form an analogous construction from an even number (6 or more) of pentagrams. (See Fig 3.11).

In this case, we deal with the edges internal to the cycle of pentagrams in such a way as to ensure that there is at least one adjacency between the regions labelled X, and also at least one adjacency between those labelled Y. By the same argument as above, if there were a region colouring, then either all the regions labelled X or all those labelled Y must take the

colour of the external region. This is a contradiction, and so again the graph so formed is a snark.

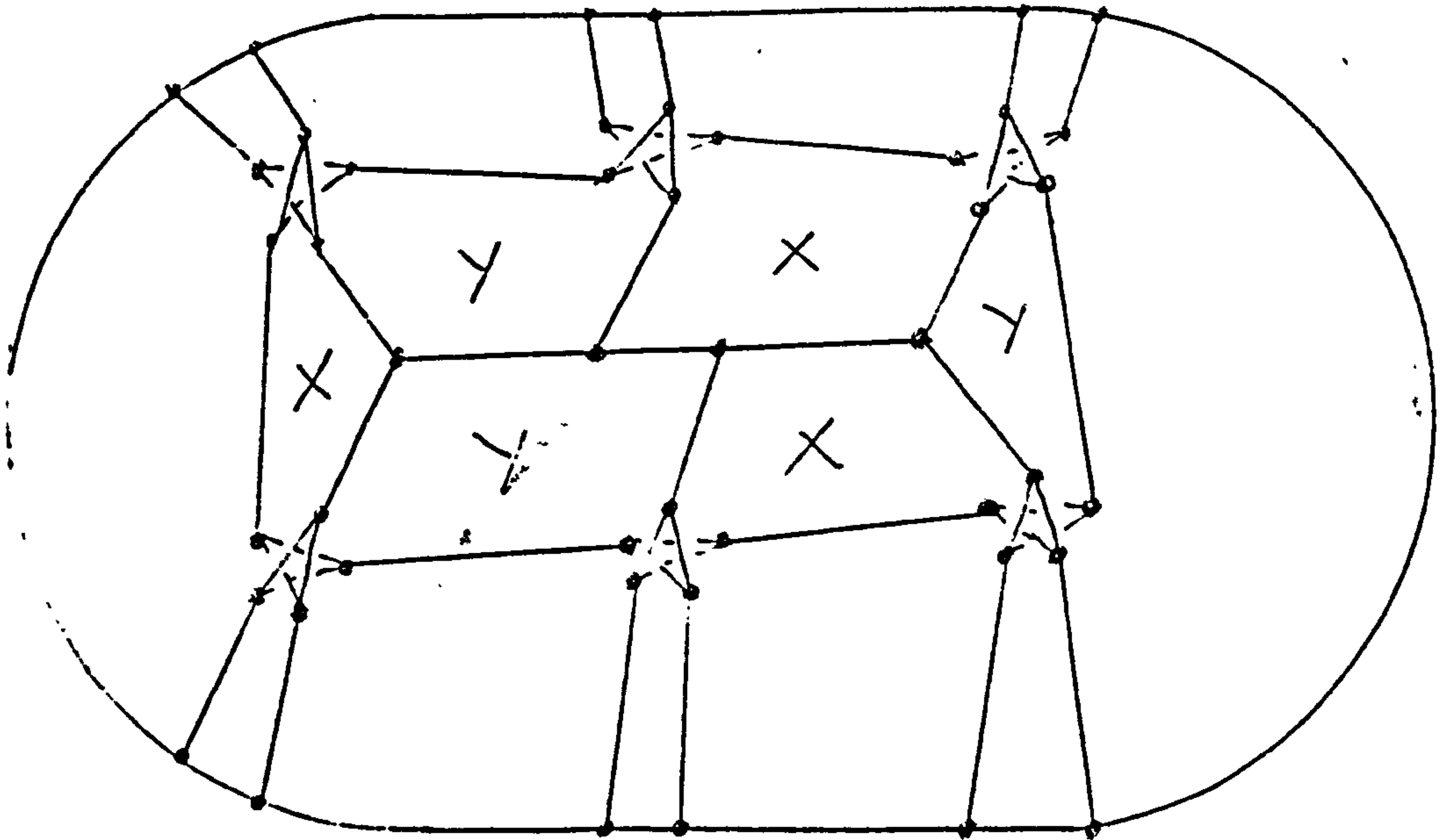


fig 3.11

CHAPTER FOUR

Heawood's Theorem and Its Extension to Angles

Let M be any p.c.b. map. We introduce the concept of a function

$$X: V(M) \rightarrow \{1, 2\}$$

which we say obeys *Heawood's congruence relations* if, for each face f of M ,

$$\sum_f X(v) \equiv 0 \pmod{3}, \tag{1}$$

the sum in (1) running over all the vertices incident with f . Such a function is known as a *Heawood labelling*; Heawood himself took the image set as $\{1, -1\}$, and in this chapter we shall use 2 and -1 interchangeably to represent the inverse of 1 in C_3 . A map with such a function has *Heawood's property*, and the conjecture that every p.c.b. map have a Heawood labelling is known as *Heawood's Conjecture*. The result that Heawood's property is equivalent to the existence of a Guthrie colouring is known as *Heawood's Theorem* [Heawood], and is proved through the intermediary of Tait's Theorem. Given a Tait colouring of M using the colours a, b, c , the Heawood label $X(v)$ is 1 if a, b and c are present with clockwise orientation at v , and is -1 otherwise.

Definition The *sense of rotation* of the Guthrie colouring of the three faces about a cubic vertex v is the same as that of the Tait colouring of the three edges incident at v , to which the Guthrie colouring corresponds. (See Fig 4.1.)

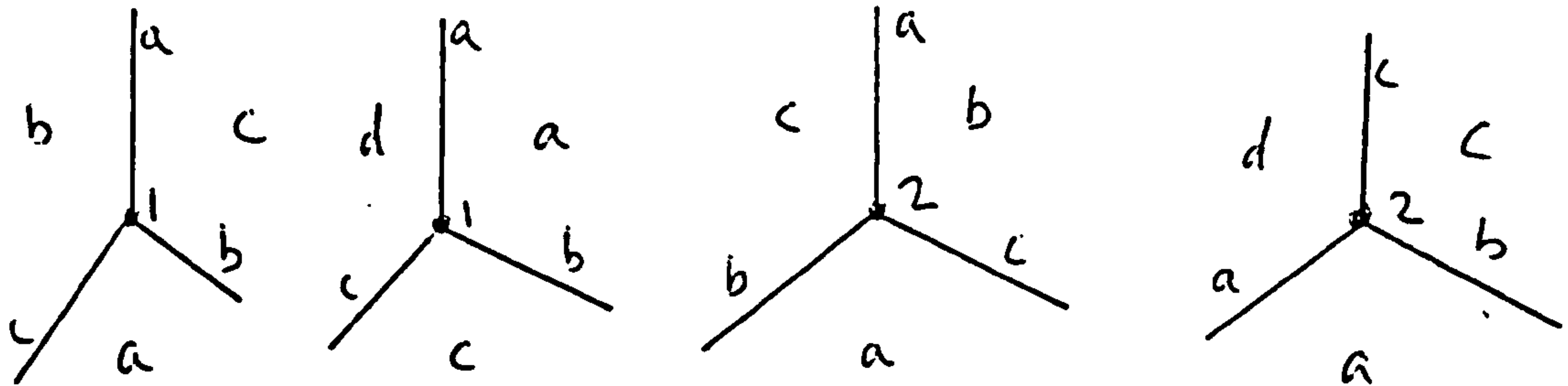


Fig 4.1

Heawood carried out extensive studies on systems (1), both with zero appearing on the right in each of the congruences and also with more arbitrary vectors.

If the map has $n+2$ faces, it has $2n$ vertices and $3n$ edges. Thus we can adjoin other equations to (1) without losing linear independence, and solve all these equations by standard linear algebra over the field of three elements. The purpose of these equations is to ensure that (1) has a solution in $X(v_1), X(v_2), \dots, X(v_{2n})$ which is nowhere zero.

Before going on to study system (1) with some non zero entries on the right hand side, let us look at one interesting case.

Angle Labellings: An Interesting Case

Suppose we can assign $X: A(M) \rightarrow \{0,1,2\}$ and $Y: A(M) \rightarrow \{0,1,2\}$ to the angles of a p.c.b. map M so that:

- (a) each value 0,1,2 appears in both X and Y labels at each vertex of M ;
- (b) $X(\alpha) \neq Y(\alpha)$ ($\alpha \in A(M)$);
- (c) for each face f ,

$$\sum_f X(\alpha) \equiv \sum_f Y(\alpha) \pmod{3},$$

the sum being taken over all angles α incident with f .

The existence of such a pair (X,Y) of labellings is equivalent to the existence of a Heawood labelling Z . For, given Z , we may choose X to take all three values round each vertex, arranged anyhow, and for each angle α :

$$Y(\alpha) = Z(v) + X(\alpha),$$

where v is the vertex incident with the angle α . Conversely, (a) and (b) ensure that $Z(\alpha) = Y(\alpha) - X(\alpha)$ takes value either 1 uniformly, or 2 uniformly, about each vertex. Condition (c) ensures that, for each face f ,

$$\sum_f Z(v) \equiv 0 \pmod{3},$$

so that Z is a valid Heawood labelling. (See Fig 4.2.)

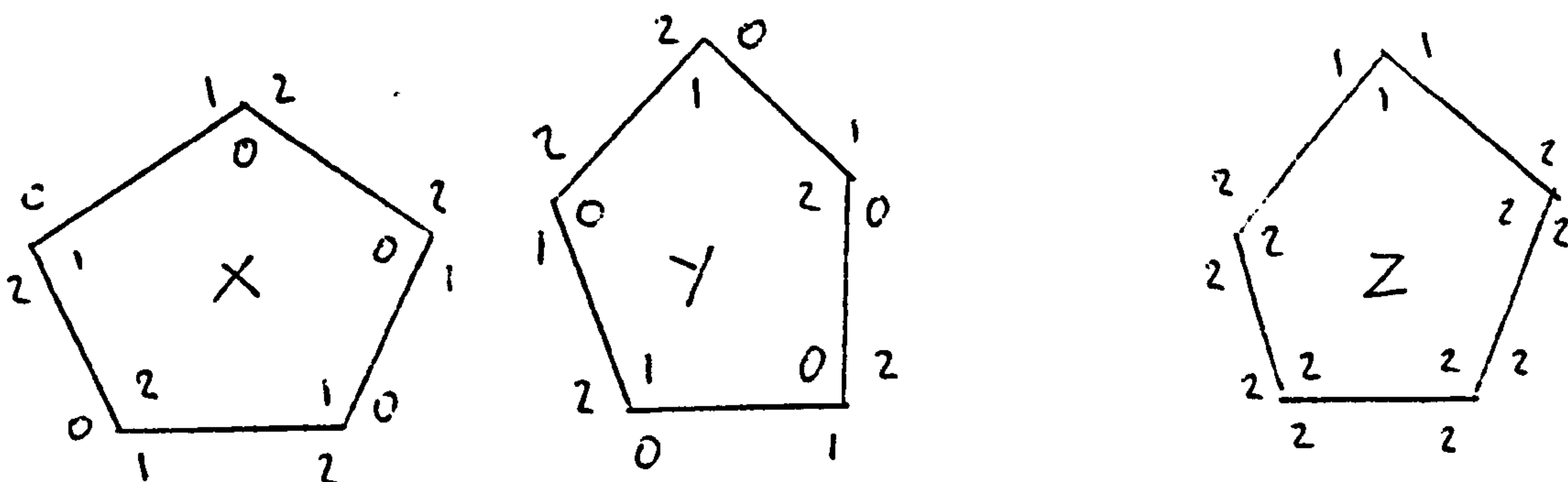


Fig 4.2

The Case of the Last Vertex

Suppose that we have a map M , for which we have found Heawood labels for all vertices except w , and that the equations (1) are obeyed for all faces except the three incident with w . Denote these faces by f_i ($i = 1, 2, 3$), and for each i let h_i be the sum (mod 3) of the Heawood labels assigned to the vertices of f_i so far. Clearly, we can assign w the Heawood label 1 or -1 in such a way as to satisfy Heawood's property, if and only if the three h_i all take the same non-zero value. The following result, an extension of a result of Heawood, shows that this is always the case.

Theorem 4.1

Let M be a p.c.b. map, let w be a vertex of M and let f_i ($i = 1, 2, 3$) be the faces containing w . Suppose that all vertices except w have been assigned non-zero Heawood labels $X(v)$, that $X(w) = 0$, and that equations (1) are obeyed for all faces of M except that

$$\sum_{F_i} X(v) = h_i \quad (i = 1, 2, 3).$$

Then $h_1 = h_2 = h_3 = 1$ or 2 .

Proof

Each vertex appears thrice in the equations (1). It follows that, whether or not the labels assigned to the vertices are non-zero, summing over all the faces gives the result $0 \pmod{3}$, so that the h_i are either all different or all the same. [Heawood 1932] has shown that the h_i cannot be all different; the argument below confirms this and shows in addition that they cannot be zero.

Let e_i ($i = 1, 2, 3$) be edges of M incident with w . Now delete w , to form a hypograph M^* with three dangling edges e_i^* ($i = 1, 2, 3$). (See Fig 4.3.)

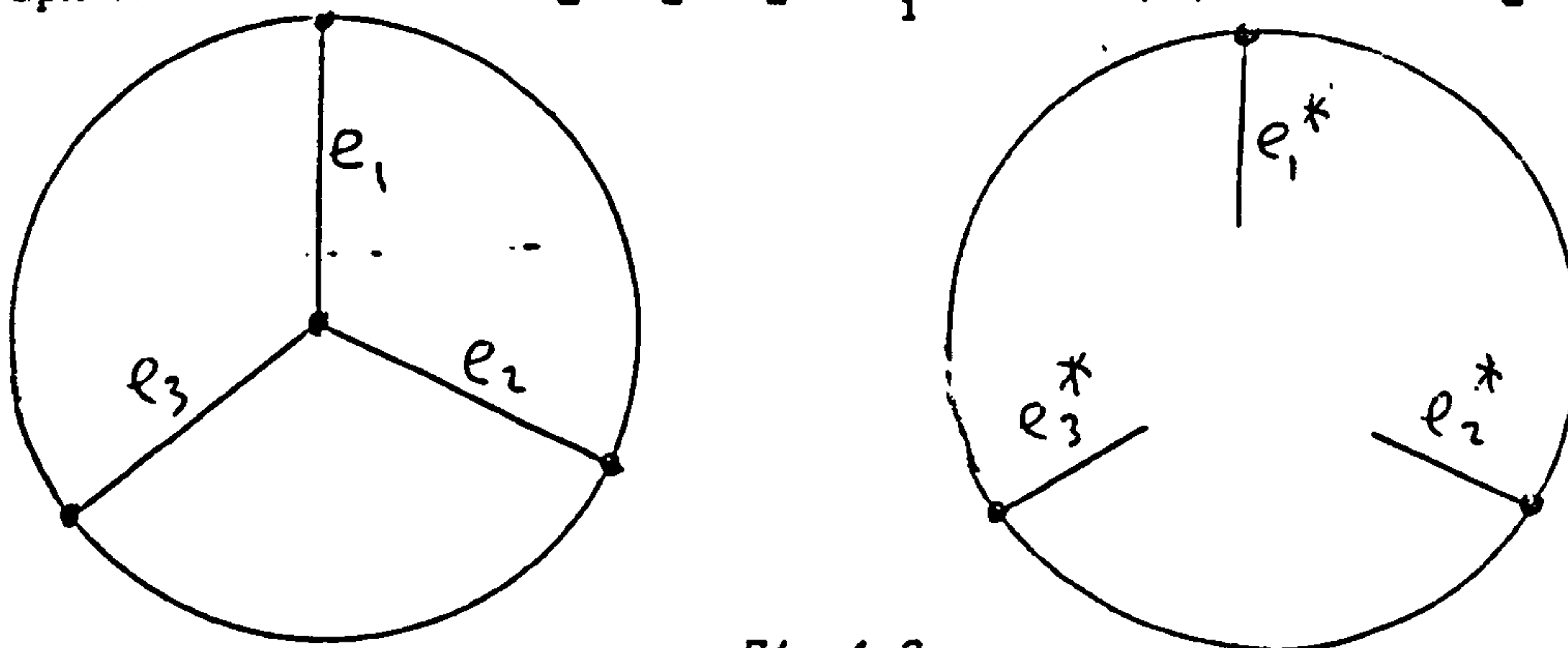


Fig 4.3

Let N^* be a copy of M^* inverted in a circle passing through the ends of the e_i^* . We insert N^* into the region where w was in M , to obtain a map G . The dangling edges e_i^* become, in G , normal edges e_i^s . (See Fig 4.4.)

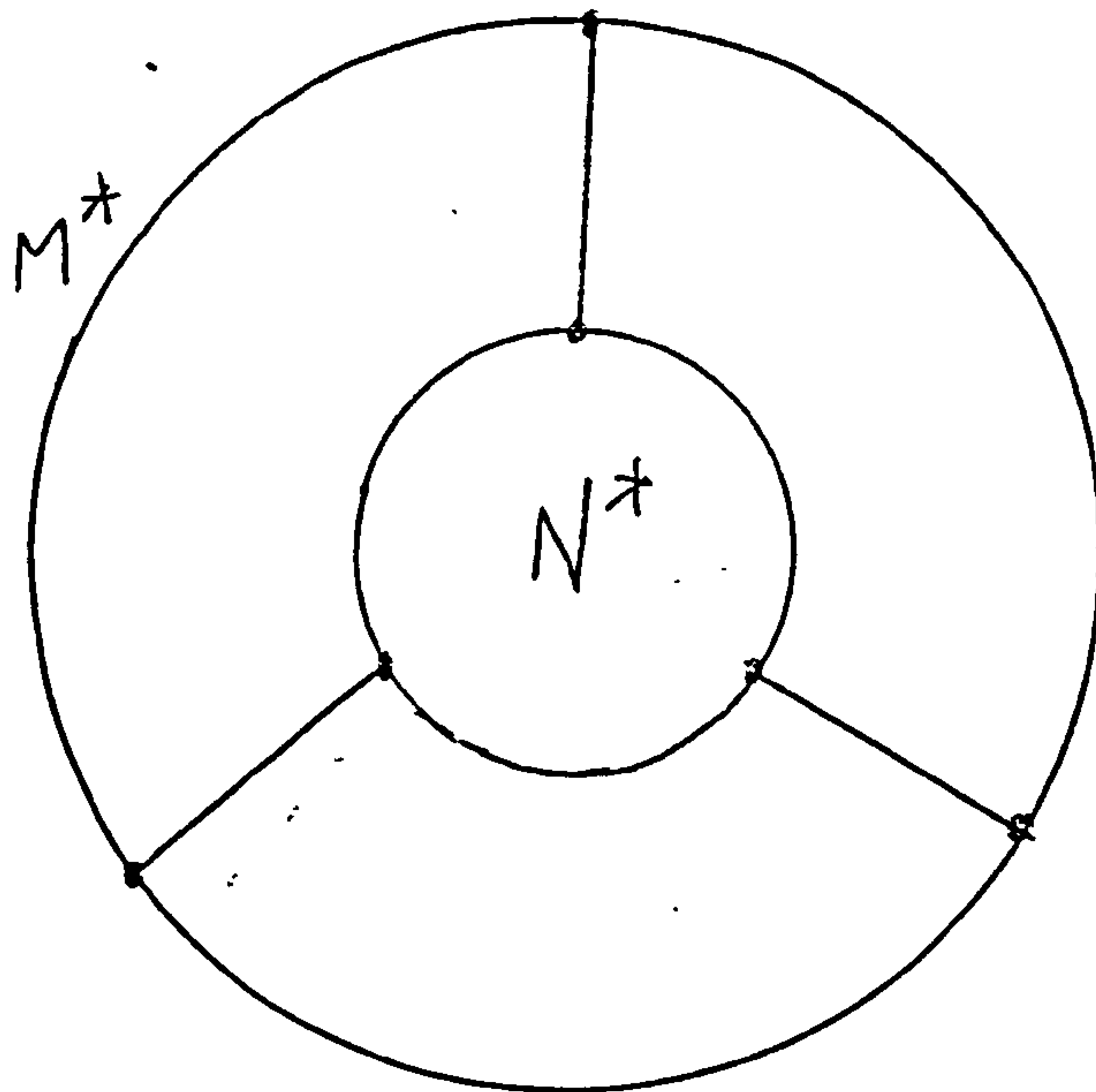


Fig 4.4

We give each vertex of N^* the Heawood label complementary to that of the corresponding vertex of M^* , producing a labelling of G that satisfies the Heawood property. By Heawood's Theorem, M has a corresponding Guthrie colouring. The edges e_1^s form a cutset, separating three faces that must receive distinct colours. We now replace N by the \bar{v} vertex w again, restricting the Guthrie colouring to M . The rotation sense of this colouring about each vertex clearly corresponds to the original Heawood labels on the vertices of M other than w . The rotation sense about w allows w to be allocated a non-zero Heawood label in such a way that the labelling of M has the Heawood property. The result follows. ■

The Technique of Angle Labelling

In the rest of this chapter we deal with the extension of Heawood's Theorem to angles in a map with either more or less than one dangling edge. Our aim is to throw some light on the four colour theorem as we work our way towards the S_3 work appearing in chapter 5.

Theorem 4.2

A p.c.b. map M is Tait colourable if and only if it has an angle labelling:

$A(G) \rightarrow \{1,2\}$ such that:

(a) the sum of the labels incident with each edge is congruent to
 $0 \pmod{2}$;

(b) the sum of the labels incident with each face is congruent to
 $0 \pmod{3}$.

Proof

See [F.Loupekine and J.J.Watkins, (2)].

Angle labelling of graphs having vertices of degree 2 and 3

Theorem 4.3

Let G be a plane graph each of whose vertices is of degree 2 or 3. Then G is edge 3-colourable if and only if it has an angle labelling:

$A(G) \rightarrow \{1,2\}$ such that:

(a) the sum of the labels incident with each face is congruent to
 $0 \pmod{3}$;

(b) the sum of the labels incident with each vertex is congruent to
 $0 \pmod{3}$.

(See Fig 4.5.)

Proof

We show that this procedure leads to an edge 3-colouring of G .

$$P = \{e_1, \dots, e_p\}$$

$\gamma(e, P)$ be the sum (mod 3) of the angle labels to the right of the path P .

If paths P and Q end at the same edge e, and differ only in going opposite ways round some face f, then condition (a) implies that $Y(e,Q) = Y(e,P)$; a

straightforward inductive argument now shows that Y depends only on e . Thus, Y defines an edge labelling Y' .

We now show that Y' obeys the condition for a colouring.

Let v be the vertex incident to any pair of adjacent edges e_1 and e_2 . If v has degree 2 then, by condition (b), one angle at v has label 1 and the other has label 2; thus whichever label is to the right in going from e_1 to e_2 , we have $Y'(e_1) \neq Y'(e_2)$. If v has degree 3, then (again by condition (b)) all the angles at v have the same label; thus, whether there is one angle or two angles on the right in going from e_1 to e_2 , we again have $Y'(e_1) \neq Y'(e_2)$.

In order to prove the converse, we assume that the edges of G have been coloured with the three colours 0, 1 and 2. Then to label the angle between two edges - say e_1 and e_2 coloured respectively with c_1 and c_2 , with e_2 counterclockwise from e_1 - we label the angle with $c_2 - c_1 \equiv 1$ or $2 \pmod{3}$. The labelling certainly satisfies condition (a), since if the edge colours clockwise around each face are c_1, c_2, \dots, c_k , then the sum of angle labels about this face is $(c_2 - c_1) + (c_3 - c_2) + \dots + (c_1 - c_k) \equiv 0 \pmod{3}$. This labelling also satisfies condition (b): at a vertex of degree 2, the two adjacent edges are coloured c_1 and c_2 say, and one angle is labelled $c_1 - c_2 \pmod{3}$ and the other $c_2 - c_1$. So one label is 1 and the other 2. At a vertex of degree 3, the three edges are coloured in anticlockwise order, either 0, 1, 2 or 0, 2, 1. In the first case the three angles are labelled 1, and in the second case the three angles are labelled 2. This completes the proof of the theorem. ■

Corollary 4.1

Let G be a plane graph each of whose vertices is of degree 2 or 3. Then G is edge 3-colourable if and only if there is a labelling with 1's and 2's of the vertices of degree 3 and of the angles at vertices of degree 2 (in

this latter case there being just one 1-label and one 2-label to each vertex), such that around each face the sum of the vertex labels (for vertices of degree 3) and the interior angle labels (for vertices of degree 2) is 0 (mod 3).

We can now generalise Theorem 4.3 to plane graphs of maximum vertex degree 3. Clearly, the presence of vertices of degree 1 and their incident edges does not affect edge 3-colourability. To see that it does not affect the existence of angle labellings satisfying conditions (a) and (b), we simply note that if we delete the vertices of degree 1 and their incident edges, to produce a graph all of whose vertices are of degree 2 or 3, and label this graph, then reinstating the missing edges and vertices is no problem: an angle labelled 1 becomes two angles labelled 2, and vice versa..

Since a vertex of degree 1 and its incident edge can be considered alternatively as a dangling edge, we have in effect generalised Theorem 4.3 to plane cubic hypographs, so that this work has a direct bearing on the general conjecture mentioned in Chapter 2.

Theorem 4.4

Let G be a planar graph each of whose vertices is of degree 2 or 3. Then G is edge 3-colourable if and only if there is an angle labelling $X: A(G) \rightarrow \{1,2\}$ such that:

(a) the sum of the angle labels incident with each face is 0 (mod 3);

(b) for each edge e joining two vertices v and w , the four angles

$\alpha_1, \dots, \alpha_4$ incident with e have labels satisfying

$$\sum_1 X(\alpha_i) \equiv \deg(v) + \deg(w) \pmod{2}.$$

Proof

First we assume that G is edge 3-colourable. Then by Theorem 4.3 there is an angle labelling such that condition (a) holds and also the sum of the labels around any vertex is $\equiv 0 \pmod{3}$. If a vertex has degree 3, therefore, all three angles have the same label, whereas, if a vertex has degree 2, then one angle is labeled 1 and the other 2. Thus, if both v and w have degree 3, then the sum over all four angles at the edge $\{v, w\}$ is 0 (mod 2); if, say v has degree 3 and w has degree 2, then the angle sum is odd; if both v and w have degree 2, then the angle sum is 6. So condition (b) holds as well.

Conversely, suppose that X is an angle labelling satisfying conditions (a) and (b). We show that G has an edge 3-colouring.

First, we construct an edge labelling $Y: E(G) \rightarrow \{0, 1\}$.

For each edge e , let v be a vertex incident with e and let α_1 and α_2 be the angles incident with e and v . Then we give e the label

$$Y(e) = X(\alpha_1) + X(\alpha_2) + \deg(v) + 1 \pmod{2}.$$

Because of condition (b), we get the same label for e if we use the angles at the other end of e . This is because

$$X(\alpha_1) + X(\alpha_2) + \deg(v) + 1 \equiv X(\alpha_3) + X(\alpha_4) + \deg(w) + 1 \pmod{2},$$

where w , α_3 and α_4 are the vertex and angles at the other end of e .

We claim that the edges labelled 1 form a disjoint set of circuits.

Let v be a vertex of degree 3, and $\alpha_1, \alpha_2, \alpha_3$ the angles incident with v . Then the sum $Y(e_1) + Y(e_2) + Y(e_3)$ for the three edges e_1, e_2, e_3 incident with v is

$$2X(\alpha_1) + 2X(\alpha_2) + 2X(\alpha_3) + 3 \deg(v) + 3 \equiv 0 \pmod{2}.$$

Thus, at a vertex of degree 3 an even number of edges are labelled 1.

Now let v be a vertex of degree 2, and α_1 and α_2 the angles incident with v . Then the sum $Y(e_1) + Y(e_2)$ for the two edges e_1, e_2 incident with v is

$$2X(\alpha_1) + 2X(\alpha_2) + 2 \deg(v) + 2 \equiv 0 \pmod{2}.$$

So, at a vertex of degree 2, both edges have the same binary label. It follows that the edges labelled 1 form a set of disjoint circuits.

By the Jordan curve theorem, we can colour the faces of G black and white so that colour differences only and always occur across these disjoint edge circuits in G (See Fig 4.6.)

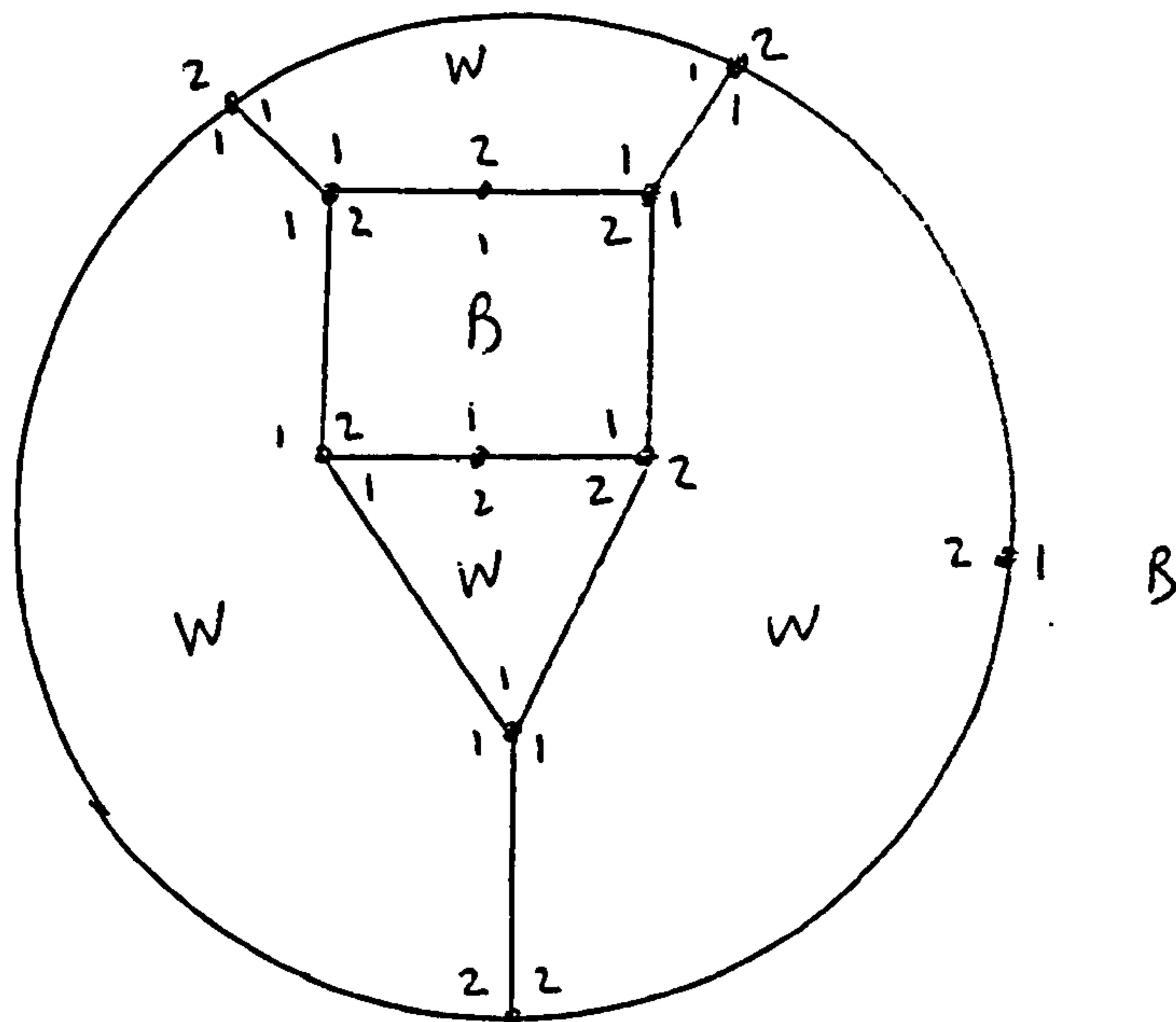


Fig 4.6

We now construct a new angle labelling X' , as follows. If an angle α is in a white region, then $X'(\alpha) = X(\alpha)$; if α is in a black region, then $X'(\alpha) = 3 - X(\alpha)$.

Condition (a) clearly holds for X' . Moreover, if α_1 and α_2 are adjacent, then $X'(\alpha_1) = X'(\alpha_2)$ if α_1 and α_2 are at a vertex of degree 3, while $X'(\alpha_1) \neq X'(\alpha_2)$ if α_1 and α_2 are at a vertex of degree 2.

Thus X' obeys the conditions of Theorem 4.3, and the result is established. ■

CHAPTER FIVE

The Symmetry Group of the Triangle

Throughout this chapter, we use the notation defined in Chapter 2; that is, a and b are regarded as elements of the symmetric group

$$S_3 = \langle a, b: a^3 = b^2 = d, ab = ba^2 \rangle,$$

in which d is the identity element.

We have seen how, in a p.c.b. map, a Tait colouring of its edges may be derived in a reversible fashion from a Guthrie colouring of the faces, with the property that four Guthrie colourings correspond to each Tait colouring. We did this by taking the colours to be elements of K , and letting each edge take the label in K which was the sum of the labels of the two faces it separated. The redundancy in the Guthrie colouring comes from the degree of choice - four - of the colour of just one initial face.

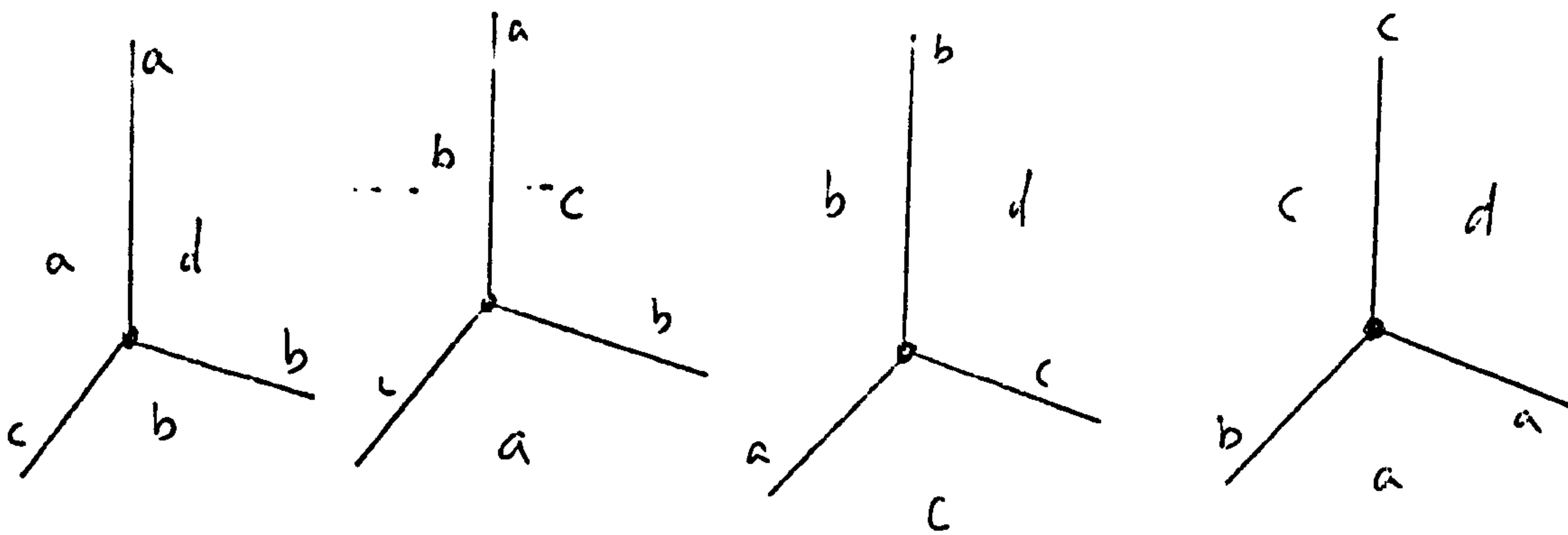
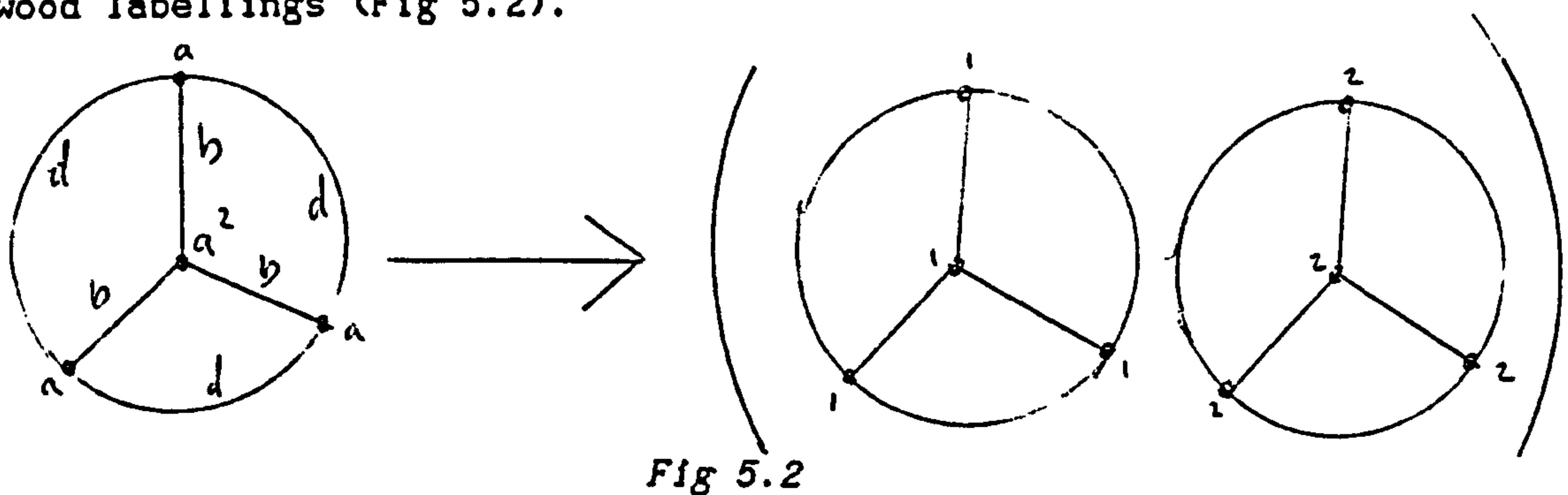


Fig 5.1

The three Tait colours about each vertex are a, b, c . If they occur in clockwise sense we assign the label 1 to the incident vertex, otherwise

we assign the label 2 (Fig 5.1). We have a Tait colouring if and only if these binary characters satisfy Heawood's C_3 congruences.

If we start with a Heawood labelling, then any choice of a , b or c for the colour of one edge allows us to construct a Tait colouring, so three Tait colourings correspond to one Heawood labelling. The process $(a,b,c,d) \rightarrow (a,b,c) \rightarrow (1,2)$ can be extended just one step further, as we shall see in Theorem 5.5, where we label the edges of G with a or b depending on whether they join vertices with similar or different Heawood labels. Thus to each of the S_3 labellings there correspond two Heawood labellings (Fig 5.2).



In this chapter we derive the S_3 labellings from what we have done on angle labellings in Chapter 4. There is no obvious way of deriving the S_3 results other than through the angle labellings of the previous chapter.

We start with three algebraic lemmas.

Lemma 5.1

If a_1, a_2, \dots, a_n are in S_3 , and

$$a_1 a_2 \dots a_n = d,$$

then for all $r < n$, we have

$$a_r a_{r+1} \dots a_n a_1 a_2 \dots a_{r-1} = d.$$

Proof

All these products are clearly in the same conjugacy class, and d is conjugate only to itself in any group. ■

Lemma 5.2

Let $A = (a_1, a_2, \dots, a_n)$ be a sequence of elements of S_3 , each equal to a , a^2 or b , with an even number (say $2k$) of occurrences of b . Let $A' = (a'_1, a'_2, \dots, a'_m)$ (where $m = n - 2k$) be the sequence formed from A by exchanging a with a^2 for each element equal to a or a^2 and with an odd number of occurrences of b to its left, then deleting all occurrences of b . For example, if $A = (a, b, a, a^2, b, a, b, a, b, a, a^2)$, then $A' = (a, a^2, a, a, a^2, a, a^2)$. Then

$$\prod_1 a_i = \prod_j a'_j.$$

Proof

Let $(b, a_j, a_{j+1}, \dots, a_k, b)$ be any subsequence of A beginning and ending with b , and with none of a_j, \dots, a_k equal to b . Since $b^2 = d$, we may replace this subsequence with $(ba_jb, ba_{j+1}b, \dots, ba_kb)$ without altering the value of the product. Thus, since $bab = a^2$ and $ba^2b = a$, the result follows. ■

Lemma 5.3

Let $A = (a_1, a_2, \dots, a_n)$ be a sequence of elements of S_3 , each equal to a or a^2 . Let $A' = (a'_1, a'_2, \dots, a'_n)$ be the sequence defined as follows. If $1 \leq i < n$, then

$$a'_i = \begin{cases} a, & \text{if } a_{i+1} = a_i, \\ b, & \text{if } a_{i+1} \neq a_i; \end{cases}$$

$$a'_n = \begin{cases} a, & \text{if } a_n = a_1, \\ b, & \text{if } a_n \neq a_1. \end{cases}$$

Then

$$\prod_1 a_i = \prod_1 a'_i.$$

Proof

Assume that $a_1 = a$. Then the process of replacing A by A' is equivalent to taking each maximal consecutive sequence of a^2 's, and replacing them and the preceding 'a' with the sequence $ba...ab$, where if there were k a^2 's, there are now $(k-1)$ a's. Since $aa^2 = d$ and $(k-1)$ a^2 's can be replaced by $ba...ab$ without affecting the product, the result follows.

A similar argument establishes the result in the case that $a_1 = a^2$. ■

Definitions

Let G be a plane graph each of whose vertices is of degree 2 or 3.

Consider two labellings $X: A(G) \rightarrow \{a, a^2\}$ and $Y: E_S(G) \rightarrow \{d, b\}$. For each face f of G , we define $P_f(X, Y)$ to be the product of the labels of the angles and edge sides incident with f , taken in cyclic order. (In general this is defined only up to conjugacy class in S_3 , but as Lemma 5.1 suggests, our interest is in the case where it takes the value d .) For each edge e of G , we define $P_e(X, Y)$ to be the total number of a 's and b 's on the two sides of e and the four incident angles, counting a^2 as two a 's.

Theorem 5.1

Let G be a plane graph each of whose vertices is of valency 2 or 3. Then G is edge 3-colourable if and only if there are labellings

$X: A(G) \rightarrow \{a, a^2\}$ and $Y: E_G(G) \rightarrow \{d, b\}$ such that:

- (a) $P_f(X, Y) = d$ for every face f of G ;
- (b) For every edge e of G , $P_e(X, Y)$ is congruent (mod 2) to the sum of the degrees of the vertices incident with e .

Proof

First we suppose that G can be edge 3-coloured. Label both sides of each edge with d . Then temporarily label the angles with 1's and 2's so that the two conditions of Theorem 4.4 hold. Now replace each angle label x by a^x ; then conditions (a) and (b) above are immediately satisfied.

Conversely, assume that G has been labelled so that conditions (a) and (b) hold. Condition (a) implies that, for every face of G , the number of incident edge sides labelled b is even.

Suppose there are $2k$ such edge sides incident with some face f . Starting at any angle of f , we go round f clockwise, exchanging angle labels a and a^2 at each angle such that we have traversed an odd number of edge sides with 'b' labels to reach it. We do this for every face, then we remove all 'b's and replace them by 'd's. By Lemma 5.2, each face still obeys condition (a), as the replacement process essentially changes a sequence A to a sequence A' as in Lemma 5.2 for each face. Moreover, on each edge side, one of three things happens:

- (1) neither the edge side label nor either of the incident angle labels changes;

- (2) the edge side label does not change, but both incident angle labels change;
- (3) the edge side label changes, and so does one of the incident angle labels.

Thus, in every case the total number of a 's and b 's at each edge is unchanged (mod 2), so condition (b) still holds. We obtain an angle labelling satisfying the conditions of Theorem 4.4 by replacing a by 1 and a^2 by 2 at all angles.

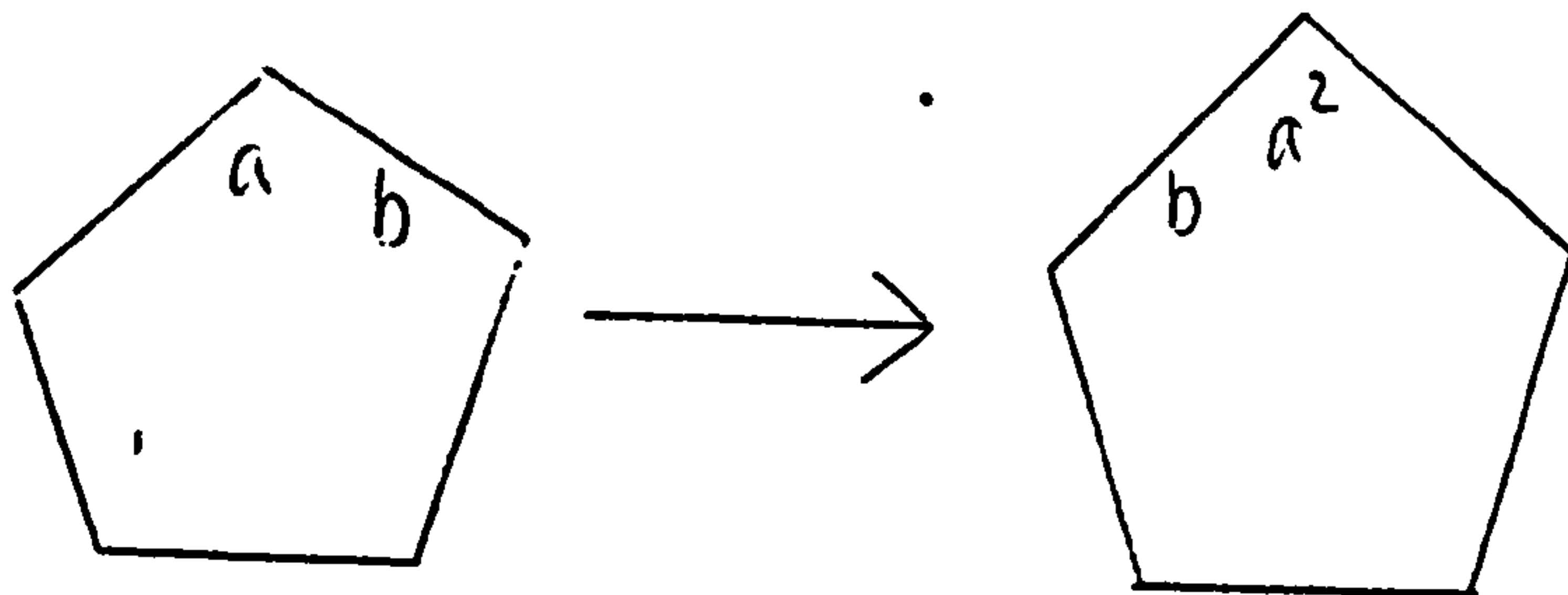


Fig 5.3

The Cubic Case

The remarks following the proof of Theorem 4.3 show that Theorems 4.3 and 4.4 may be regarded as applying to plane 3-regular hypographs with k dangling edges, in which case they assume particularly simple forms in the case $k = 0$. Theorem 5.1 also has a useful special case, as follows (see Fig 5.4):

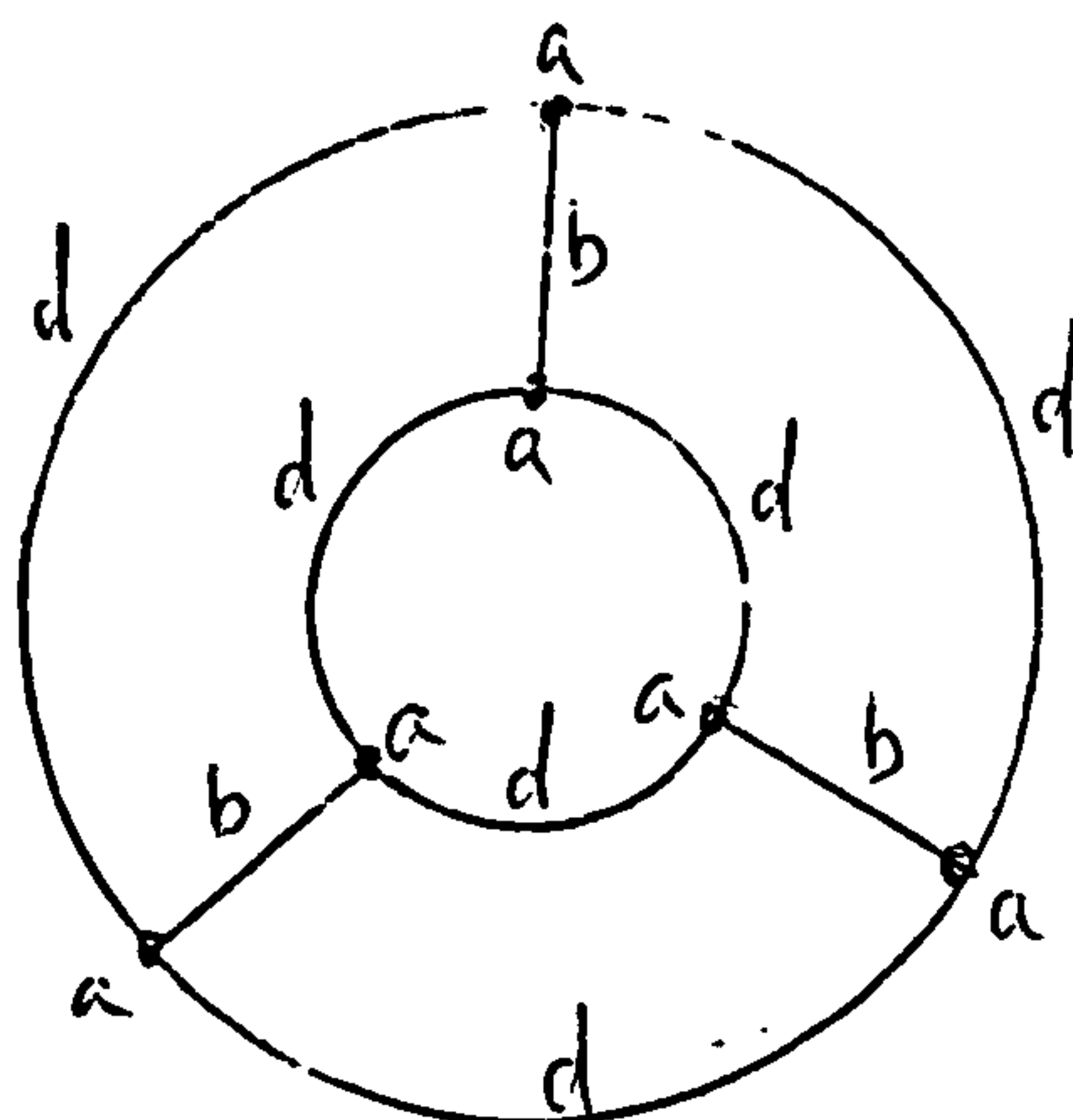


Fig 5.4

Theorem 5.2

Let G be a p.c.b. graph. Then G is Tait colourable if and only if there are labellings $X: A(G) \rightarrow \{a, a^2\}$ and $Y: E_G(G) \rightarrow \{d, b\}$, so that:

- (a) $P_f(X, Y) = d$ for every face f of G ;
- (b) $P_e(X, Y) \equiv 0 \pmod{2}$ for every edge e of G .

We now generalise Theorem 5.1 in a different direction, as follows.

Theorem 5.3

Let g, h be any elements of S_3 . Then if (X, Y) is any pair of labellings obeying condition (a) of Theorem 5.1, the pair (gXh^{-1}, hYg^{-1}) obtained by replacing each angle label x by gxh^{-1} and each edge side label y by hyg^{-1} also satisfies this condition.

Proof.

Let $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ be the angle and edge labels in order round a face. Then:

$$\begin{aligned}
 \prod_1 x_1 y_1 = d & \iff \prod_1 (g^{-1} g) x_1 (h^{-1} h) y_1 = d \\
 & \iff g \left(\prod_1 (g^{-1} g) x_1 (h^{-1} h) y_1 \right) g^{-1} = d \\
 & \iff \prod_1 (g x_1 h^{-1}) (h y_1 g^{-1}) = d.
 \end{aligned}$$

There are thus many different labellings of angles and sides of edges, all obeying conditions equivalent to the conditions (a) and (b) of Theorem 5.1. We just let h and g run over the whole of S_3 .

We now generalize the P_f notation which we used in Theorems 5.1 and 5.2. First, if X is a vertex labelling and Y an edge labelling of a plane graph, then for each face f of G , we define $P_f(X, Y)$ to be the product of the vertex and edge labels incident with f , taken in cyclic order. (As before, this is defined up to conjugacy class in S_3 .) Next, letting D denote the edge (or vertex) labelling with all labels equal to d , we define $P_f(X) = P_f(X, D)$ and $P_f(Y) = P_f(D, Y)$.

Theorem 5.4

A p.c.b graph G is Tait colourable if and only if there are labellings $X: V(G) \rightarrow \{d, a\}$ and $Y: E(G) \rightarrow \{a, b\}$ such that

$$P_f(X, Y) = d$$

for each face f of G .

Proof

Let G be a p.c.b. graph for which we can find labellings X and Y obeying the condition of the theorem. We now construct angle and edge side labellings X' and Y' as follows.

For each angle α incident with vertex v , we define

$$X'(\alpha) = a^2 X(v) a^2,$$

and for each side s of edge e , we define

$$Y'(s) = a Y(e) a.$$

The properties of S_3 imply that $X': A(G) \rightarrow \{a, a^2\}$ and

$Y': E_S(G) \rightarrow \{d, b\}$. Moreover, using Theorem 5.3 with $g = a^2$, $h = a$, we

conclude that X' and Y' obey condition (a) of Theorem 5.1. Condition (b) is clearly obeyed, and so by Theorem 5.1, G is Tait colourable.

Conversely, suppose G is Tait colourable. Then, by Heawood's Theorem, it has a Heawood labelling, H , of its vertices. We now construct an angle labelling X' and an edge side labelling Y' , as follows.

If α is an angle incident with vertex v , then $X'(\alpha) = a^{H(v)}$.

For each edge side s , $Y'(s) = d$.

The fact that $a^3 = d$ in S_3 immediately implies that X' and Y' obey the conditions of Theorem 5.1. We now use Theorem 5.2 with $g = a$ and $h = a^2$, to replace a with d and a^2 with a in the angles, and d with a on the edge sides, to obtain the required labellings X and Y . ■

Theorem 5.5

A p.c.b graph G is Tait colourable if and only if it has an edge labelling $Y: E(G) \rightarrow \{a, b\}$ such that

$$P_f(Y) = d$$

for each face f of G .

Proof

First, suppose that Y is an edge labelling obeying the condition of the theorem. Set $X = D$; then $P_f(X, Y) = d$ for each face f of G , and so G is Tait colourable by Theorem 5.4.

Conversely, suppose that G is Tait colourable. Then it has a Heawood vertex labelling H . Define the vertex labelling X by

$$X(v) = a^{H(v)}, \quad v \in V(G).$$

Thus, $P_f(X) = d$ for each face f of G . Now define the edge labelling Y by

$$Y(\langle u, v \rangle) = \begin{cases} a, & \text{if } X(u) = X(v), \\ b, & \text{if } X(u) \neq X(v). \end{cases}$$

If $A = (a_1, \dots, a_n)$ is the sequence of vertex labels round some face, then the sequence A' , derived from A as in Lemma 5.3, is the corresponding sequence of edge labels. Thus $P_f(Y) = d$ for each face f of G , as required. ■

Definition

Let G be a p.c.b. graph, with labellings $X: V(G) \rightarrow \{a, a^2\}$ and $Y: E(G) \rightarrow \{d, b\}$. Then X and Y are *compatible* if

$$P_f(X, Y) = d$$

for each face f of G .

Theorem 5.6

Let G be a Tait colourable p.c.b. graph. Then for every vertex labelling $X: V(G) \rightarrow \{a, a^2\}$, there is a compatible edge labelling $Y: E(G) \rightarrow \{d, b\}$.

Proof

Let $X: V(G) \rightarrow \{a, a^2\}$, and let H be a Heawood vertex labelling (which exists, as G is Tait colourable). Define the vertex labelling

$$Z(v) = a^{H(v)}, v \in V(G).$$

Now define an edge labelling Y as follows.

$Y(\langle u, v \rangle) = d$ if either $X(u) = Z(u)$ and $X(v) = Z(v)$ or $X(u) \neq Z(u)$ and $X(v) \neq Z(v)$;

$Y(\langle u, v \rangle) = b$ if either $X(u) \neq Z(u)$ and $X(v) = Z(v)$ or $X(u) = Z(u)$ and $X(v) \neq Z(v)$.

Thus, as we go round any face f of G , the edges labelled b separate vertex sequences on which X and Z agree from vertex sequences on which they disagree. If there are no such edges incident with f , then either $X(v) = Z(v)$ for each vertex v of f , in which case $P_f(X, Y) = P_f(Z) = d$, or $X(v) = (Z(v))^{-1}$ for each vertex v of f , giving $P_f(X, Y) = (P_f(Z))^{-1} = d$. On the other hand, if there is at least one edge e of f with $Y(e) = b$, then there is some vertex v_1 of f such that $X(v_1) = Z(v_1)$.

Now let $A = (a_i)$ be the sequence of alternating vertex X labels and edge Y labels obtained by going round f starting at v_1 , so that $P_f(X, Y) = \prod_1 a_i$. If we delete the occurrences of d in the sequence, then A is as described in Lemma 5.2, and the corresponding A' is just the sequence $(Z(v_1), Z(v_2), \dots)$. Thus, by Lemma 5.2,

$$P_f(X, Y) = P_f(Z) = d$$

for each face f of G , as required. ■

Theorem 5.7

Let G be a Tait colourable p.c.b. graph. Then for every edge labelling $Y: E(G) \rightarrow \{d, b\}$ such that $P_f(Y) = d$ for each face f , there is a compatible vertex labelling $X: V(G) \rightarrow \{a, a^2\}$.

Proof

Let Y be such a labelling. Then there are an even number of edges labelled b in any face of G , and hence in any circuit of G . Thus we may colour the vertices of G black and white so that each edge labelled d joins similarly coloured vertices and each edge labelled b joins oppositely coloured vertices. Let Z be a vertex labelling constructed as in the proof of Theorem 5.6, and let X be the labelling:

$V(G) \rightarrow \{a, a^2\}$ which agrees with Z on black vertices and disagrees with Z on white vertices. Then $\langle X, Y \rangle$ bears exactly the same relation to Z as in the proof of Theorem 5.6, and so we conclude that

$$P_f(X, Y) = d$$

for each face f of G , as required. ■

Summary

Heawood's property for a p.c.b. map can be restated as follows: It is possible to assign a, a^2 to vertices so the product around each face is the identity in S_3 .

We have seen in Theorem 5.6 that whatever the vertex labels are, we can assign edge labels d, b so that this identity still holds. We see further that it is not always possible to extend an edge labelling with d, b to a corresponding vertex labelling with a, a^2 . A necessary and sufficient condition is that each face have an even quota of b 's.

CHAPTER SIX

Kleinian Colourings in Kleinian Graphs

That the two colourings a p.c.b graph may have, namely a Guthrie colouring and a Tait colouring, always occur together, has been shown by [Tait]. A Tait colouring may be viewed as a colouring in the Klein group K , with the following two restrictions :

- (a) The sum of edge labels in each cutset is d , the identity in K .
- (b) Adjacent edges have different labels.

Lemma 6.1

Let G be a plane graph, having an edge labelling X with elements of K . Then X obeys condition (a) if and only if, for each vertex v of G , the sum of the edge labels incident with v is d .

Proof

The edges incident with any vertex form a cutset, and so the sufficiency of the condition is clear.

Conversely, suppose the sum of labels incident with each vertex is d , and let C be a cutset, separating the vertex set W from $V(G) - W$. The sum of the labels incident with each vertex of w is d , and each edge except those in C is counted twice. Since K is Abelian and $x^2 = d$ ($x \in K$), the result follows. (See Figure 6.1.) ■

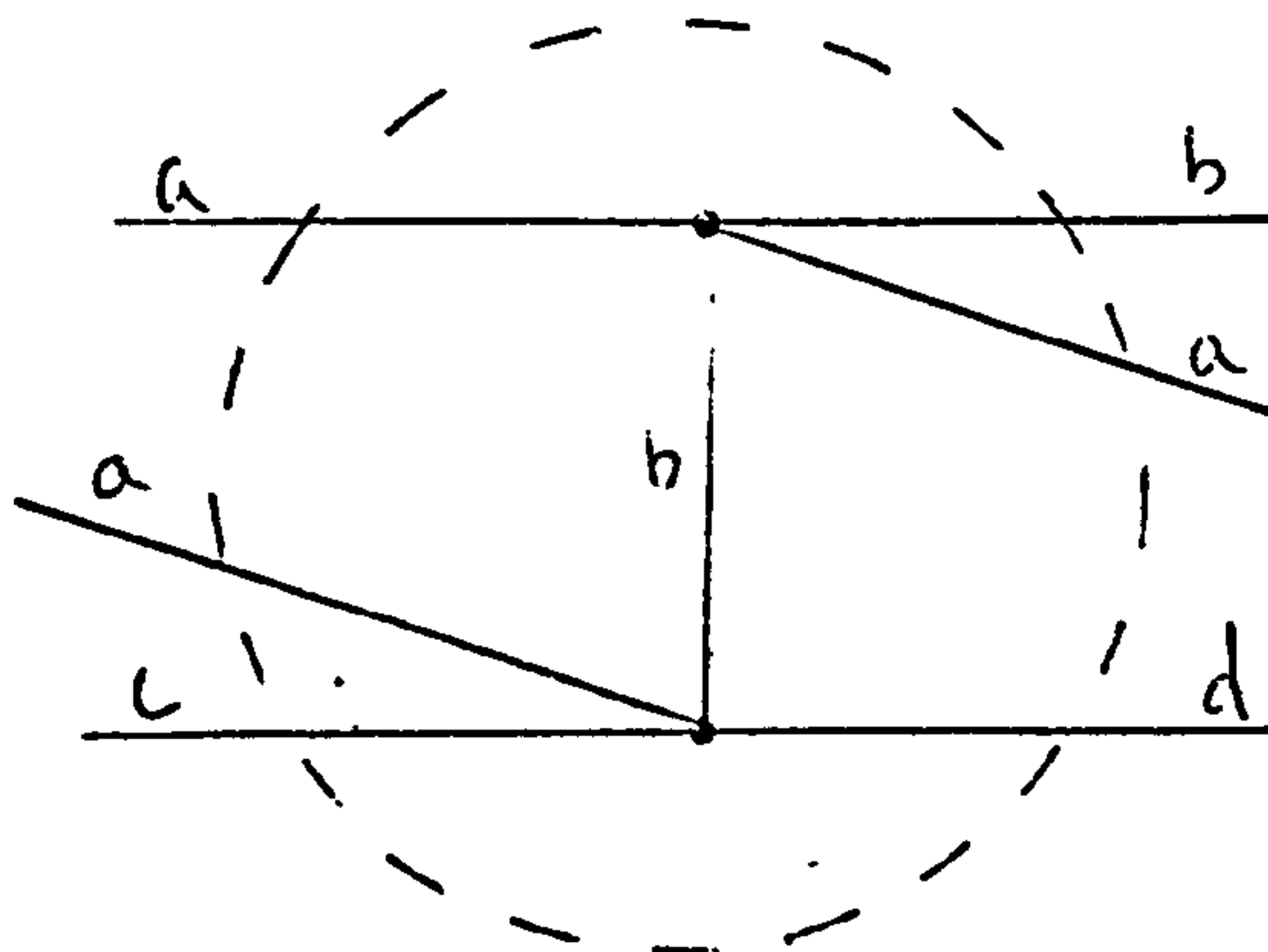


Fig 6.1

In a cubic graph, condition (a) implies that the edge labels incident with a vertex constitute one of the sets:

$$\{d,d,d\}, \{d,a,a\}, \{d,b,b\}, \{d,c,c\}, \{a,b,c\}.$$

Only one of these trios has a permutation which obeys condition (b) above, and that is $\{a,b,c\}$. Thus a labelling of a cubic graph obeying both these conditions is a Tait colouring using the non-identity elements of K as colours.

In considering plane graphs other than cubic ones, we shall make use of the notion of *geometrical adjacency*.

Definitions

Two edges of a plane graph G are *geometrically adjacent* if they are incident with a common angle (see Fig 6.2).

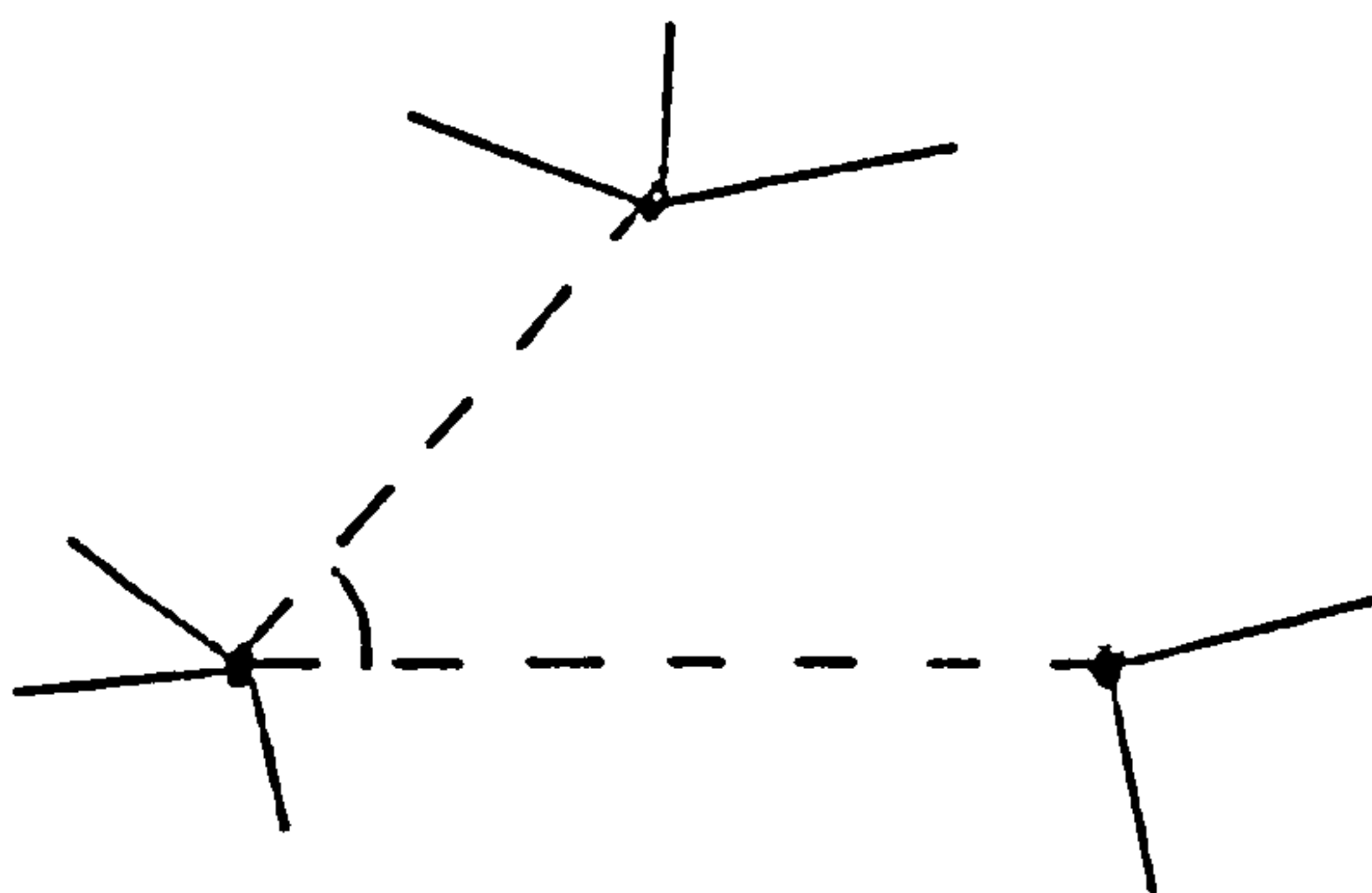


Fig 6.2

An edge labelling $X: E(G) \rightarrow K$ is said to be a *Kleinian colouring* if:

(a) for each vertex v of G ,

$$\sum_v X(e_i) = d, \text{ the sum being taken over the edges incident with } v;$$

(b) $X(e_1) \neq X(e_2)$ whenever e_1 and e_2 are geometrically adjacent.

We say that a plane graph is *Kleinian* if it has a Kleinian colouring.

Theorem 6.1

No plane graph with a vertex of degree 2 is Kleinian.

Proof

It is clearly impossible for conditions (a) and (b) to be obeyed at such a vertex. ■

Theorem 6.2

No connected plane graph containing vertices only of degrees 1 and 3, with at least one vertex of each of these degrees, can be Kleinian.

Proof

Such a graph must contain an edge e incident with a vertex v of degree 1 and also with a vertex w of degree 3. Condition (a) at v implies that $X(e) = d$, but the labels at w must be a, b, c . This is a contradiction. ■

Theorem 6.3

An edge labelling of a planar quartic graph with elements of K is Kleinian if and only if the edge labels about each vertex are either:

(1) all different, or

(2) just two colours present, opposing edges taking identical colours. (See Fig 6.3.)



Fig 6.3

Proof

The only lists of colours satisfying condition (a) for a Kleinian colouring are:

- (i) $\{x, x, x, x\}$; for any $x \in K$;
- (ii) $\{x, x, y, y\}$, for any distinct $x, y \in K$;
- (iii) $\{a, b, c, d\}$.

Since condition (b) must also be satisfied in a Kleinian colouring, the result follows. ■

The Medial Graph

Definition

Let G be a plane graph. Then the *medial graph*, G_S , is the graph whose vertices are in 1-1 correspondence with the edges of G , two vertices of G_S being adjacent if and only if the corresponding edges in G are geometrically adjacent. Thus G_S is the result of the following two operations (see Fig 6.4):

- (1) joining with a new edge the mid points of each pair of geometrically adjacent edges of G ;

(2) erasing all of G except the mid points of its edges, which become the vertex set of G_S .

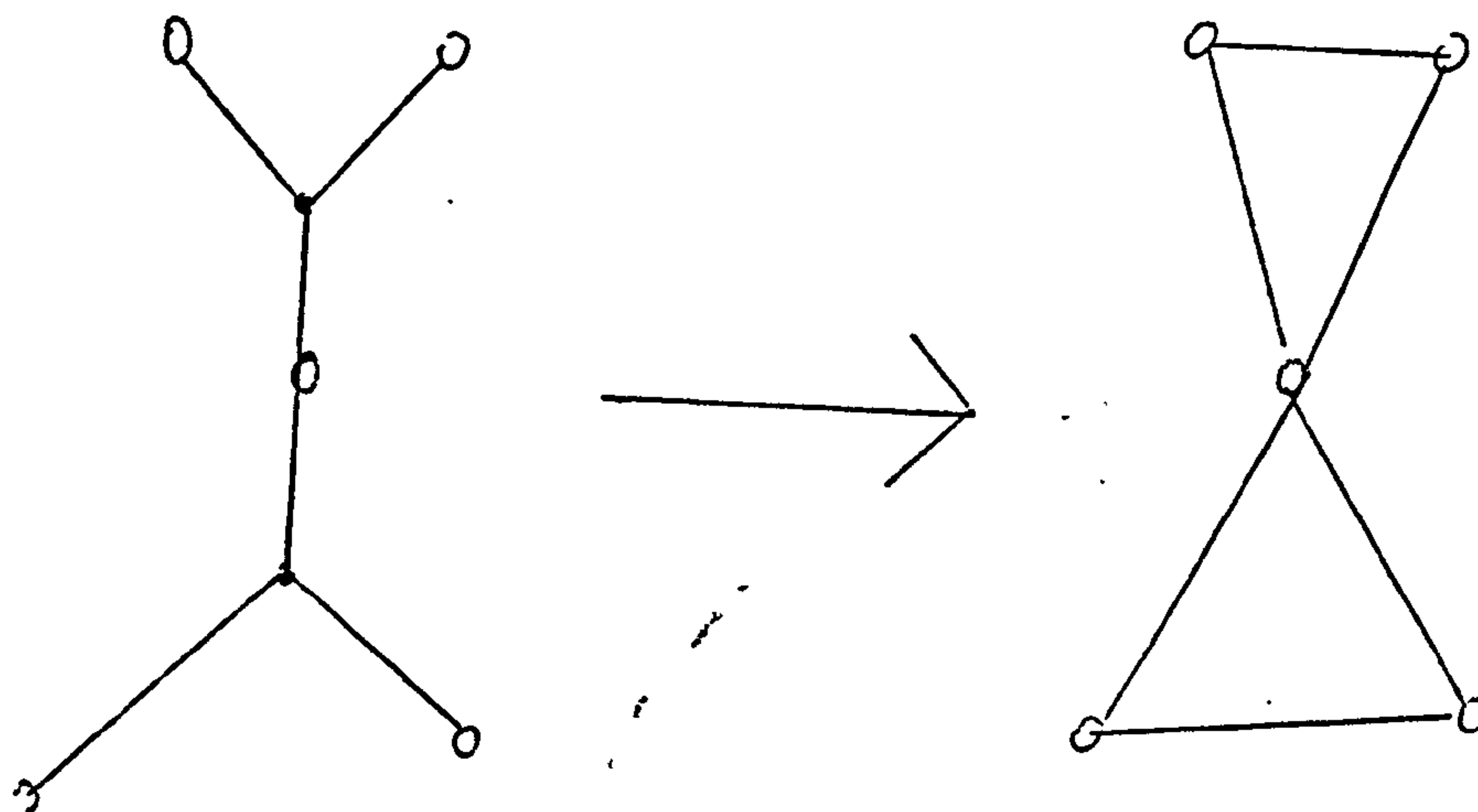


Fig 6.4

Two Kinds of Face in G_S

Let G be a plane graph. Then G_S has two sorts of face, namely those corresponding to vertices of G , which are called V -faces, and those arising from the faces of G , called F -faces. (See Fig 6.5.)

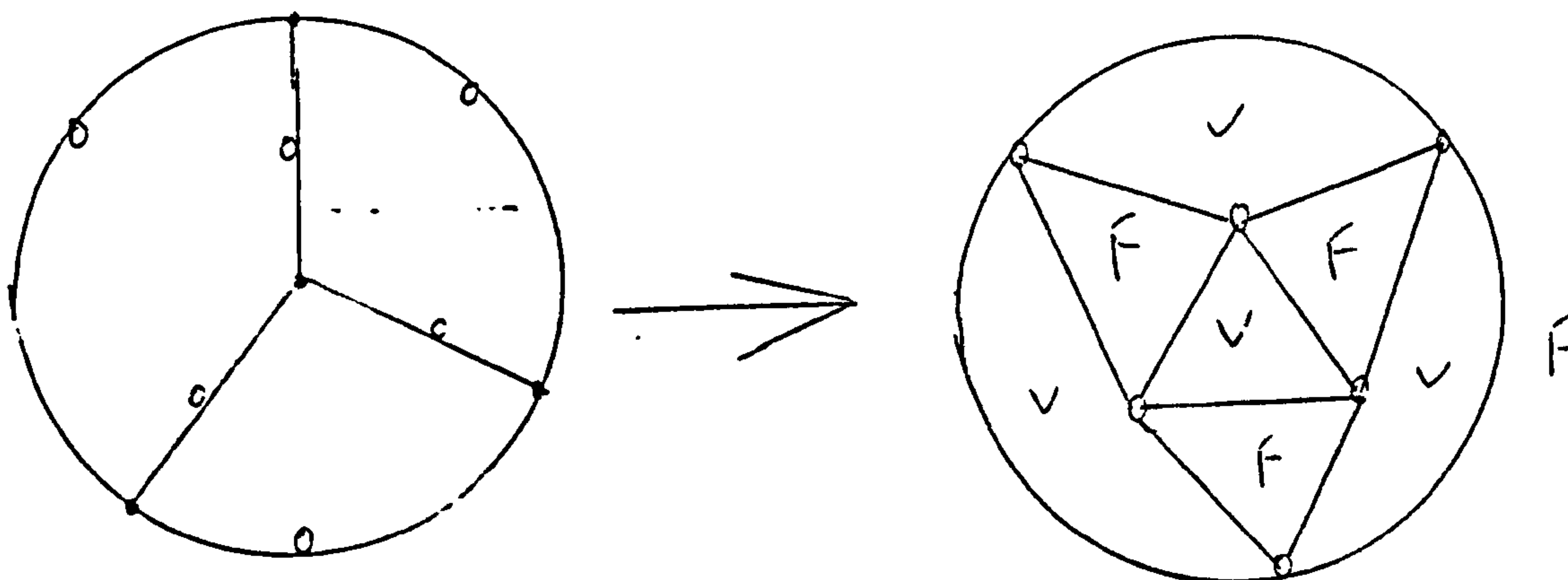


Fig 6.5

Theorem 6.4

Let G be a connected plane graph. There is a 1 - 4 correspondence between the set of all Kleinian colourings of GS and the Cartesian product of:

(a) the set of proper vertex 4-colourings of G , and

(b) the set of Guthrie colourings of G .

Proof Given a Kleinian colouring X of GS , we construct a labelling Y of the faces of GS , as follows.

We select an arbitrary face f of GS and let $Y(f)$ be any element of K . (This arbitrary choice gives the 1 - 4 correspondence.) We now define the labels of the remaining faces by requiring that, whenever two faces f_1 and f_2 are separated by an edge e , then

$$Y(f_2) = Y(f_1) + X(e).$$

The fact that G is connected implies that Y is unique. It is well-defined since, by Lemma 6.1, whenever we follow a closed path that crosses only edges of GS and not vertices, the sum of the edge labels which we encounter is d .

Now let Y_1 and Y_2 be the restrictions of Y to the V -faces and the F -faces of GS respectively. Clearly, these define labellings of $V(G)$ and $F(G)$ respectively, using the elements of K . The fact that these labellings are colourings follows from the fact that two vertices (or faces) of G are adjacent if and only if there is a path between the corresponding faces of GS , that crosses two geometrically adjacent edges of GS . Condition (b) of the definition of a Kleinian colouring implies that these two vertices (or faces) of G must receive different colours.

Thus there is a 1 - 4 correspondence between the Kleinian colourings of GS and the pairs (vertex 4-colouring of G , Guthrie colouring of G) that can be

formed in this way. The fact that all such pairs of colourings are accounted follows immediately from the fact that any such pair defines a pair (Y_1, Y_2) of labellings of the V- and F-faces of GS , which in turn define a labelling Y which is clearly Kleinian. ■

Theorem 6.5.

The statement that all 3-connected regular plane graphs have Kleinian colourings is equivalent to the Four Colour Theorem.

Proof

A 3-connected regular graph must have degree at least 3, and a regular plane graph must have degree at most 5. Thus, the only degrees to be considered are 3, 4 and 5.

In the cubic case, Kleinian colourability is equivalent to Tait colourability, as we saw at the beginning of this chapter.

In the quartic case, we note that any 3-connected plane quartic graph G is a map, whose face set may be partitioned into two sets V and F such that each face in V is adjacent only to faces in F and vice versa. Thus, G is the medial graph of a plane graph M , and also of its dual. Thus, by Theorem 6.4, G has a Kleinian colouring if and only if M and its dual have vertex 4-colourings.

There is just one plane connected 5-regular graph, namely the net of the icosahedron, which has the following Kleinian colouring. (Fig 6.6.)

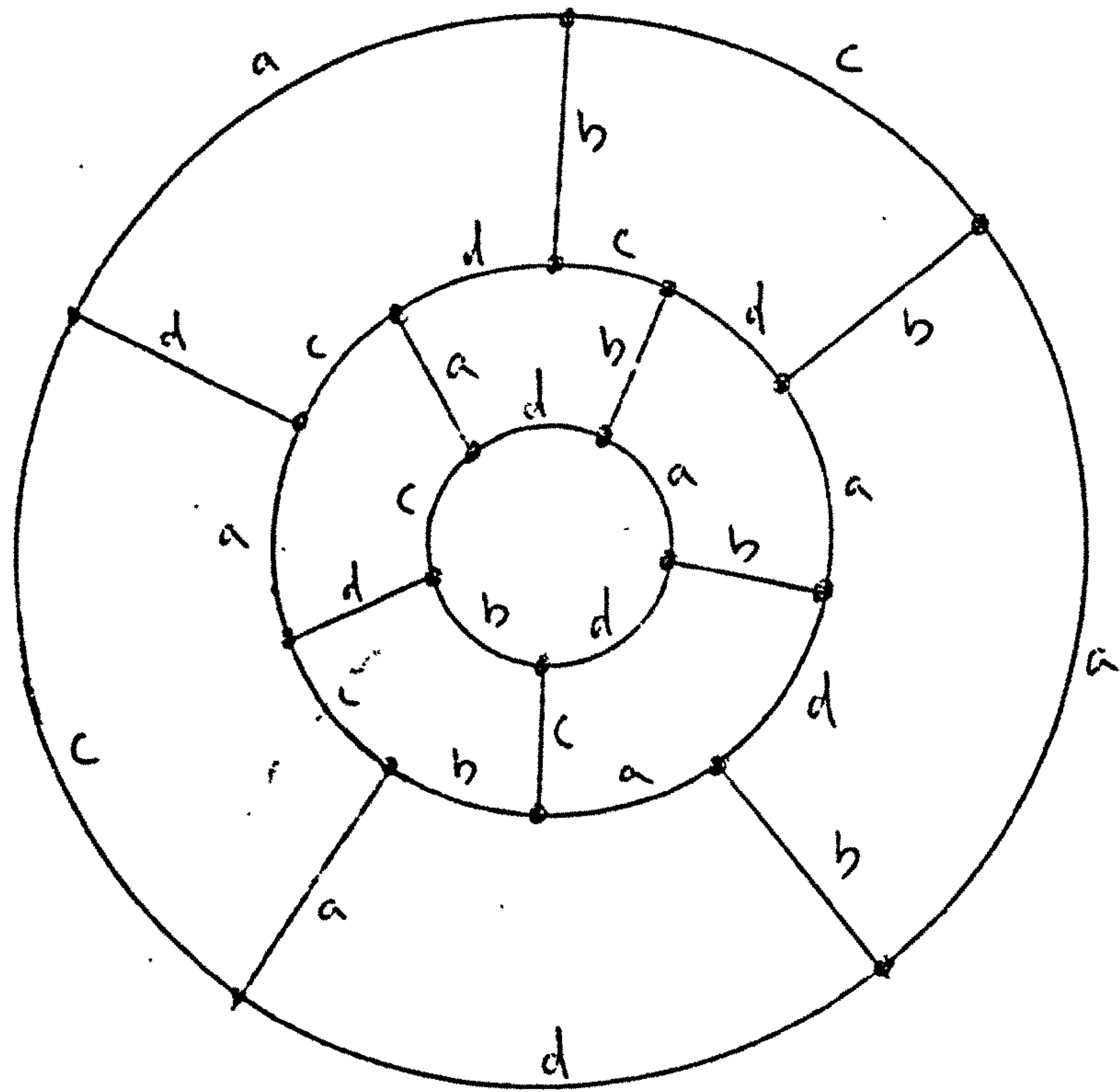


Fig 6.6

CHAPTER SEVEN

Odd Cut Samples, Odd Cycle Samples

Sample theory is a new branch of mathematics, and we shall do a little to advance this subject here.

Definition

Given a collection Ω of non-empty subsets of a set E , then a *sample* of Ω is a subset S of E such that, for every set $C \in \Omega$, $S \cap C$ is a non-empty proper subset of C .

Consider the process of finding a bound for the chromatic number of a graph G with vertex set V , by partitioning $E(G)$ into k disjoint sets S_1, S_2, \dots, S_k , thereby defining k graphs with vertex set V . For each $i = 1, \dots, k$, let f_i be a colouring of the graph (V, S_i) ; then the labelling (f_1, \dots, f_k) of V with k -tuples of colours is clearly a colouring of G , since if $(v, w) \in S_i$, then $f_i(v) \neq f_i(w)$. Thus,

$$\chi(G) \leq \prod_1 \chi(V, S_i).$$

This is not a very sharp result. It can be used, for example, to show that a Hamiltonian plane graph has chromatic number at most 8, by letting S_1 be the set of edges of the Hamiltonian circuit (adding an extra vertex if necessary to make the number of vertices even), then letting S_2 and S_3 be the sets of edges on either side of the Hamiltonian circuit. (See Fig 7.1.)

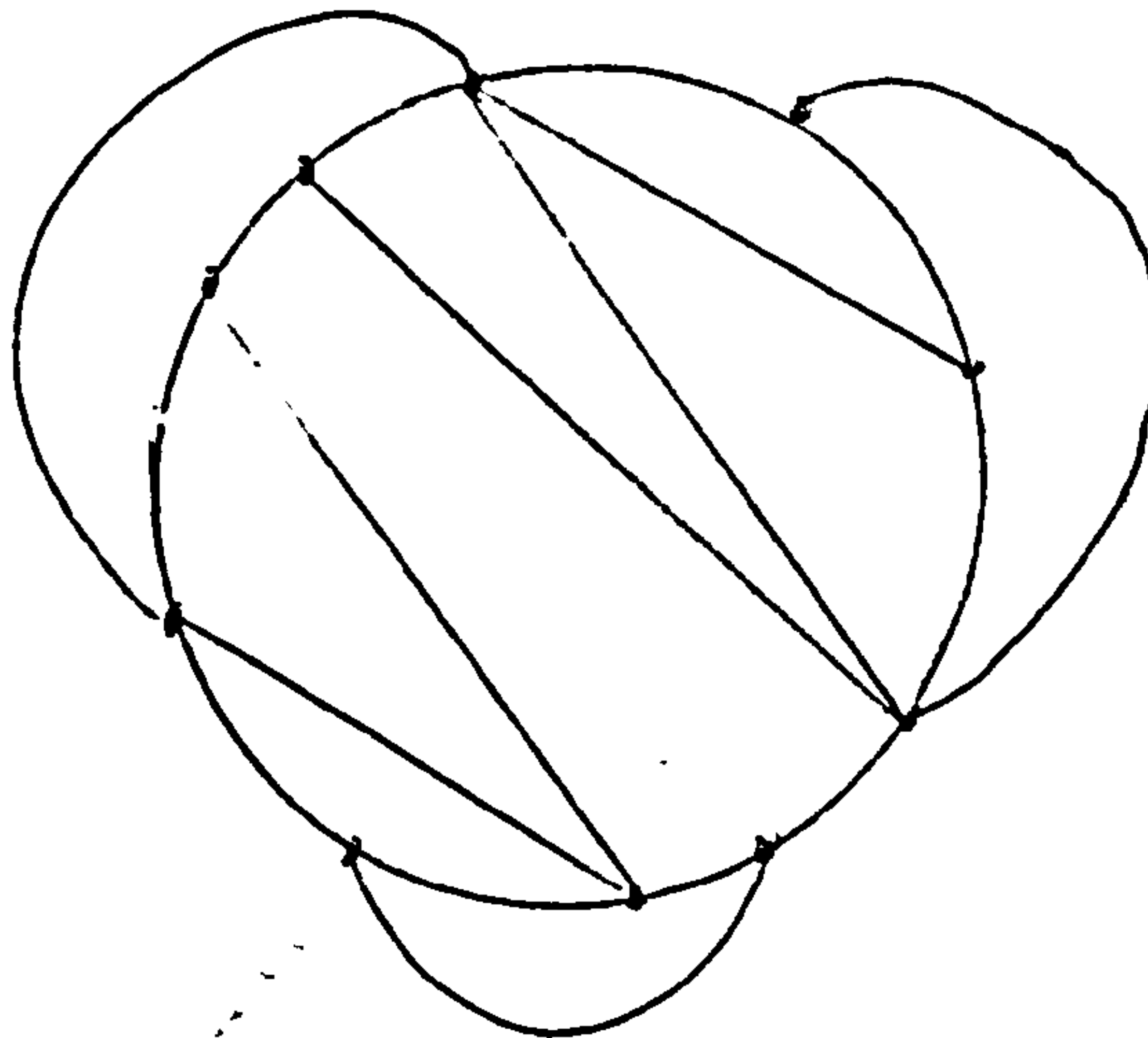


Fig 7.1

There are, of course, better bounds for the chromatic number of a plane graph! However, if $E(G)$ can be partitioned into two sets neither of which contains an odd circuit, then we have a bound of 4 on $\chi(G)$. This motivates the following definition.

Definition

Let G be a graph. An *odd circuit sample* of G is a subset of $E(G)$ which is a sample of the set $\Omega(G)$ of odd circuits of G .

Theorem 7.1

A graph G has chromatic number at most 4 if and only if it has an odd circuit sample.

Proof

Suppose first that $\chi(G) = 4$. Let $f: V(G) \rightarrow \{a, b, c, d\}$ be a proper vertex 4-colouring. Then we define S by:

$$S = \{e = \{u, v\}: \{f(u), f(v)\} = \{a, b\} \text{ or } \{c, d\}\}.$$

Clearly, S contains no odd circuit. But the complement of S is a bipartite graph, the two vertex sets being $f^{-1}(\{a, b\})$ and $f^{-1}(\{c, d\})$. Thus S intersects every odd circuit in a non-trivial proper subset, and so is an odd circuit sample.

Conversely, let S be an odd circuit sample, and T its complement in $E(G)$. Since S and T are odd circuit samples, the graphs $(V(G), S)$ and $(V(G), T)$ are each of chromatic number at most 2. Thus, $\chi(G) \leq 4$. ■

Theorem 7.2. Let M be any p.c.b. map. Then it is possible to find a subset S of $E(M)$ such that:

- (a) for each face F of M , $|S \cap E(F)| \equiv |E(F)| \pmod{2}$;
- (b) no pair of edges in S are adjacent.

Proof

$E(M)$ obeys (a). Therefore there is a set R of edges of minimum size (possibly empty) obeying (a). We shall see that R also obeys (b). Suppose not. Then there are adjacent edges r, s in R , each incident with some vertex v . Let t be the third edge incident with v . Note that t may be in R . If so, let $R' = R - \{r, s, t\}$. If not, let $R' = R \cup \{t\} - \{r, s\}$. Then R' satisfies (a) and contradicts the minimality of R . ■

Odd Cut Samples

We shall now concentrate on odd cut samples. We shall be following closely the first part of [Holroyd and Loupekinel]. It is of interest that Eckhard Steffen (private communication) has pointed out counterexamples to our *Bottleneck Conjecture*. They include the Isaac flower snarks J_k , $k > 5$.

The following definition of an odd cut is due to Lovasz and Plummer, but we have certain reservations as it does not deal with the case when G has odd components.

Definition

Let S be any graph each of whose components is of even order. Then an *odd cut* of G is a minimal set of edges C of G whose removal would result in a graph having two components of odd order.

Lemma 7.1

Let C be an odd cut in a graph G , all of whose vertices have odd degree. Then C has an odd number of edges.

Proof

Let K be one of the odd components of $G-C$. Let s be the sum of the vertex degrees of K in G , and let t be the sum of the vertex degrees of K in $G-C$. Then s is odd and t is even. But each edge in C is incident with exactly one vertex of K . Thus $|C| = s-t$. ■

Lemma 7.2

The following two conditions on a set S of edges of a connected graph G of even order, are equivalent:

- (1) Every component of the subgraph $(V(G), S)$ is of even order.
- (2) S has non-zero intersection with every odd cut of G .

Proof

Let S obey condition (1) and let C be any odd cut of G . Denote by K and L the two components of $G-C$. Then at least one component of the subgraph $(V(G), S)$ must have vertices in both K and L . Thus C intersects S , and so condition (2) is obeyed.

Conversely, Suppose S does not obey condition (1). Then some component Q of $(V(G), S)$ is of odd order. The set of edges linking vertices of Q to vertices not in Q is an odd cut that does not intersect S , and so condition (2) is not obeyed. ■

We shall call an edge set obeying these conditions an *even component generator*, or equivalently an *odd cut intersector*.

Definition

Let Ω be the set of odd cuts in a graph G of even order. Then an *odd cut sample* is a subset of $E(G)$ which is a sample of Ω .

Lemma 7.3

A subset S of $E(G)$ is an odd cut sample if and only if S and $E(G)-S$ are odd cut intersectors.

Proof

This follows at once from the definition. ■

Theorem 7.3

The following two conditions on a set S of edges of a connected cubic graph are equivalent:

- (1) S is an odd cut sample.
- (2) S and $E(G)-S$ each contain a 1-factor in G .

Proof

Let S be an odd cut sample. Since the set of edges incident with any vertex is an odd cut, it follows that each vertex of $V(G)$ is incident with at least one edge of S . Since this is also true of $E(G)-S$, each vertex of G is incident with one or two edges of S . Thus, the components of $(V(G), S)$ are paths and circuits. By Lemma 7.2, they are of even order. It follows that S contains a 1-factor of G . By the same argument, so does $E(G)-S$.

Conversely, suppose that S and $E(G)-S$ both contain 1-factors of G . It follows that every component of $(V(G), S)$, and every component of $(V(G), E(G)-S)$ is of even order. Hence, by Lemmas 7.2 and 7.3, S is an odd cut sample. ■

Corollary 7.1

A connected cubic graph G is Tait colourable if and only if it has an odd cut sample.

Proof

Suppose that G has a Tait colouring. As this partitions $E(G)$ into three 1-factors, the existence of a large number of odd cut samples is readily apparent.

Conversely, let S be an odd cut sample. Then S and $E(G)-S$ each contain a 1-factor (say, F and H respectively). Thus $E(G) - (F \cup H)$ is a third 1-factor, giving a Tait colouring. ■

We note that Corollary 7.1 can also be deduced from Theorem 7.1, since G is Tait colourable if and only if it is Guthrie colourable, and hence if and only if its dual is vertex 4-colourable; and the dual of an odd cut is an odd circuit. However, the connection with 1-factors is not so explicit in this derivation.

CHAPTER EIGHT

Alternate Angle Labelling

Following on from the work in Chapter 6 on labelling with elements of the Klein group, we proceed to develop an angle labelling with binary labels 0,1. By a *loop* we mean a cycle of neighbouring angles, and a Kleinian labelling might be constructed by sampling angles in these loops. Typically the angles of a planar quartic graph are covered by a mere handful of loops, and this makes the corresponding handful of samples easy to find. Indeed it may always be possible to cover the angles in very few loops. Maybe all 4 connected quartic planar maps can be covered in just the smallest possible number of loops, namely two.

Definition

Two angles in a plane quartic graph are *edgewise adjacent* if they are incident to a common edge but lie at opposite ends of that edge. See Fig 8.1, where angle α_1 is edgewise adjacent to α_3 and α_4 , but not to α_2 .

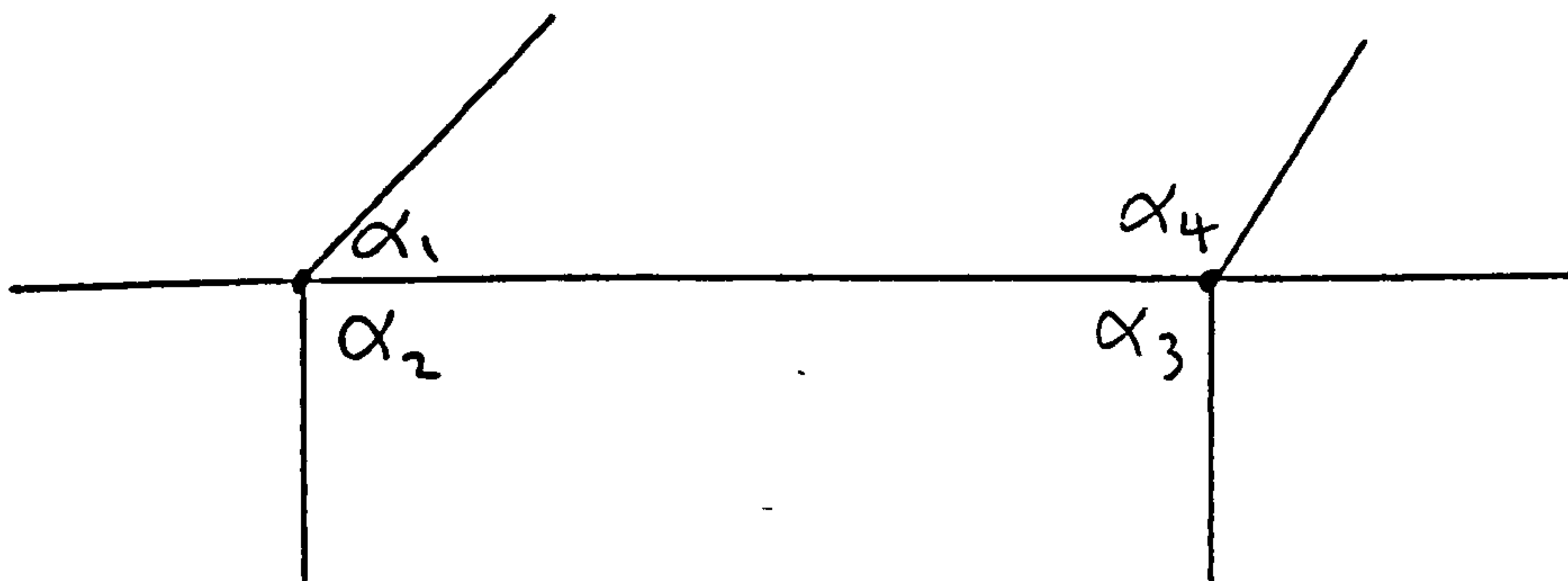


Fig 8.1

Definition

A *loop* is a sequence $\alpha_0, \alpha_1, \dots, \alpha_n$ of distinct angles such that:

(a) α_i is edgewise adjacent to α_{i+1} ($i = 0, 1, \dots, n-1$) and α_n is edgewise adjacent to α_0 ;

(b) the edge between α_i and α_{i+1} is distinct from the edge between α_{i+1} and α_{i+2} ($i = 0, 1, \dots, n-2$), and the edge between α_{n-1} and α_n is distinct from the edge between α_n and α_0 .

Definition

An *alternate labelling* of the angles of a plane quartic graph is one in which the two labels 0 and 1 alternate around each vertex. (See Fig 8.2.)

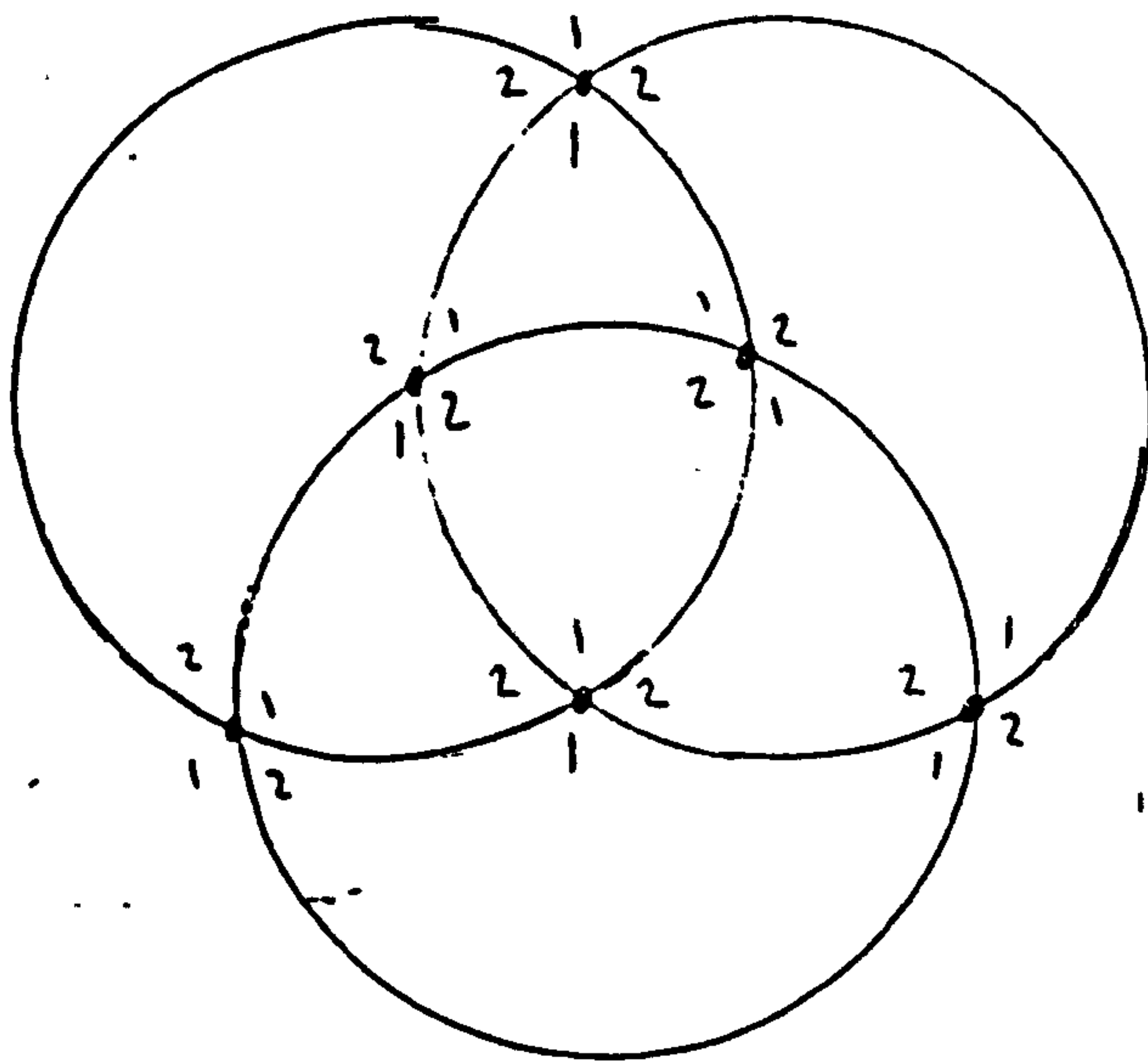


Fig 8.2

The existence of alternate labellings is trivial. We now show, however, that there is a further property which we may require an alternate labelling to possess, namely Alternate Angle labelling, or A.A. for short. This guarantees the existence of a Kleinian colouring of the graph.

Definition

An i -loop ($i = 0, 1$) in a plane quartic graph, with an alternate labelling, is a loop all of whose angles are labelled i .

Although the definition of a loop does not exclude the possibility that some edge of the graph may be traversed twice (once in each direction) in proceeding round the loop, the alternation of labels round each vertex excludes this possibility for a 0-loop or 1-loop. It is nevertheless possible for two opposite angles at the same vertex to belong to the same 0-loop or 1-loop. If this occurs, then we may visualise each such vertex as being detached into two vertices of order 2, so that the edges and vertices of the loop now form a simple closed curve. (See Fig 8.3.)

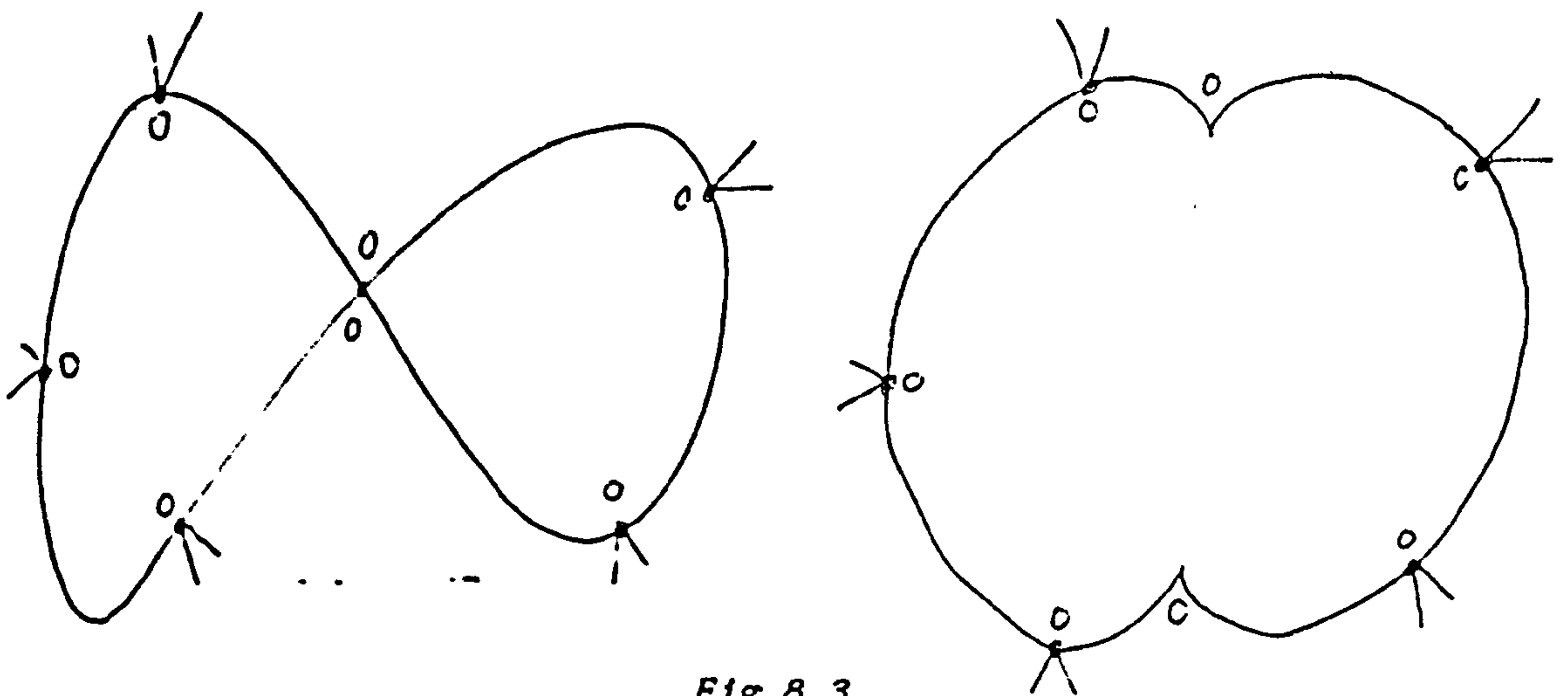


Fig 8.3

Thus, each 0-loop and 1-loop has a well-defined anticlockwise sense of traversal.

Definition

An alternate labelling in a quartic graph is said to be an *Alternate Angle (A.A.)* labelling if each 0-loop and each 1-loop is of even length.

Definition

Given a quartic plane graph, an A.A. labelling of its angles, and a particular 0-loop or 1-loop Λ , the *turning label* of any angle of Λ is L or R according as we turn left or right at that angle in traversing Λ in anticlockwise sense. (See Fig 8.4.)

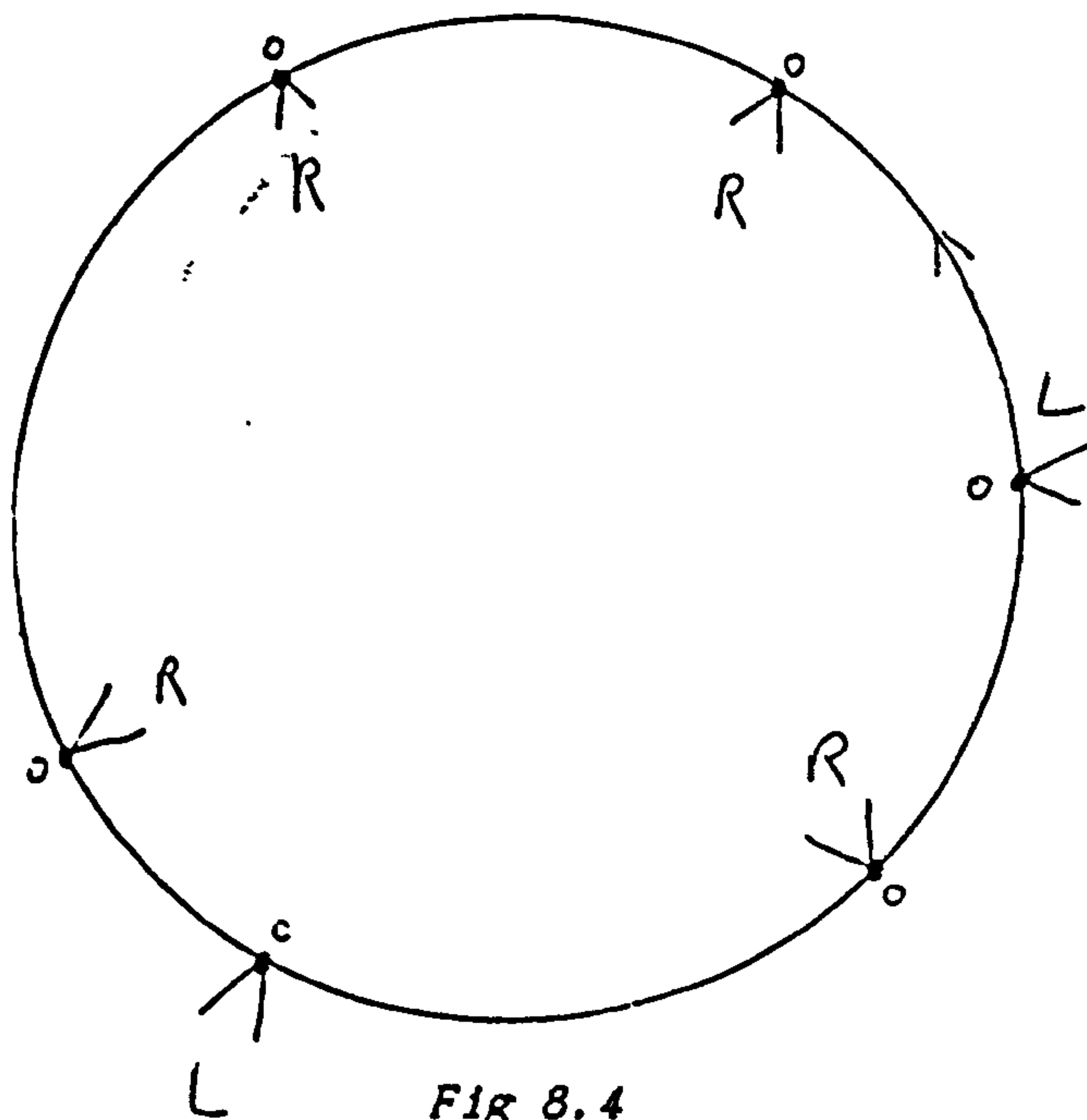


Fig 8.4

Theorem 8.1

A plane quartic map G has an A.A. labelling whenever it has a vertex labelling $Q: V(G) \rightarrow \{L, R\}$ such that the turning labels of each odd loop agree with Q on at least one vertex and disagree with Q on at least one other vertex.

Proof

Let Q be as in the statement of the theorem. As G is quartic, it has a proper face colouring with the colours B, W. Now define an A.A. labelling as follows: an angle takes the label 0 if it is in a face coloured B and at a vertex labelled R, or in a face coloured W at a vertex labelled L; otherwise the angle takes the label 1. It is easily verified that each

0-loop and each 1-loop either agrees with Q on all its vertices, or disagrees with Q on all its vertices. Thus none of these circuits can be odd, and we have an alternate angle labelling. ■

Theorem 8.2

If a plane quartic graph G has an A.A. labelling, then it has a Klein colouring.

Proof

Let Q be an A.A. labelling. We define binary functions g and h on the edges of G by requiring that the g values alternate on the edges of each 0-loop, while the h values alternate on the edges of each 1-loop. More precisely, let α, β, γ be three consecutive angles in a 0-loop [resp. 1-loop], let e be the edge between α and β , and let f be the edge between β and γ . Then $g(e) \neq g(f)$ [resp. $h(e) \neq h(f)$]. The fact that the 0- and 1-loops are all of even length implies this labelling is well defined, and there is a choice of just two labellings per loop. Either will do.

Since every angle belongs to a 0-loop or a 1-loop, it follows that for any geometrically adjacent pair e, f of edges, it cannot be true that $g(e) = g(f)$ and $h(e) = h(f)$. Also, since there are two angles labelled 0 and two labelled 1 at each vertex, two of the edges e_i incident with each vertex have $g(e_i) = 0$ and two have $g(e_i) = 1$, and the same is true of h . Thus the pairs $(g(e), h(e))$ define a Kleinian colouring of the edges of G . ■

Corollary 8.1

If the medial graph of a p.c.b. map M has an A.A. labelling, then M has a Guthrie colouring.

Of course, Theorem 8.2 gives the motivation for studying Alternate Angle labellings. The trouble in Theorem 8.1 arises at odd loops. If we could

find a sparse set of edges such that the rest of the graph is even, then we might be able to define a set of turning labels that is R everywhere except on this sparse set. We shall see in the corollary to Theorem 8.3 that we can indeed find such a sparse set in the cubic case. But our goal eludes us in the more important quartic case.

Theorem 8.3

Given any graph G , there exists a (possibly empty) subset H of $E(G)$ so that (see Fig 8.5):

(a) for each circuit C , $|H \cap C| \equiv |C| \pmod{2}$;

(b) for each vertex v , $|H \cap E_v| \leq \frac{1}{2}|E_v|$, where E_v is the set of edges incident with v .

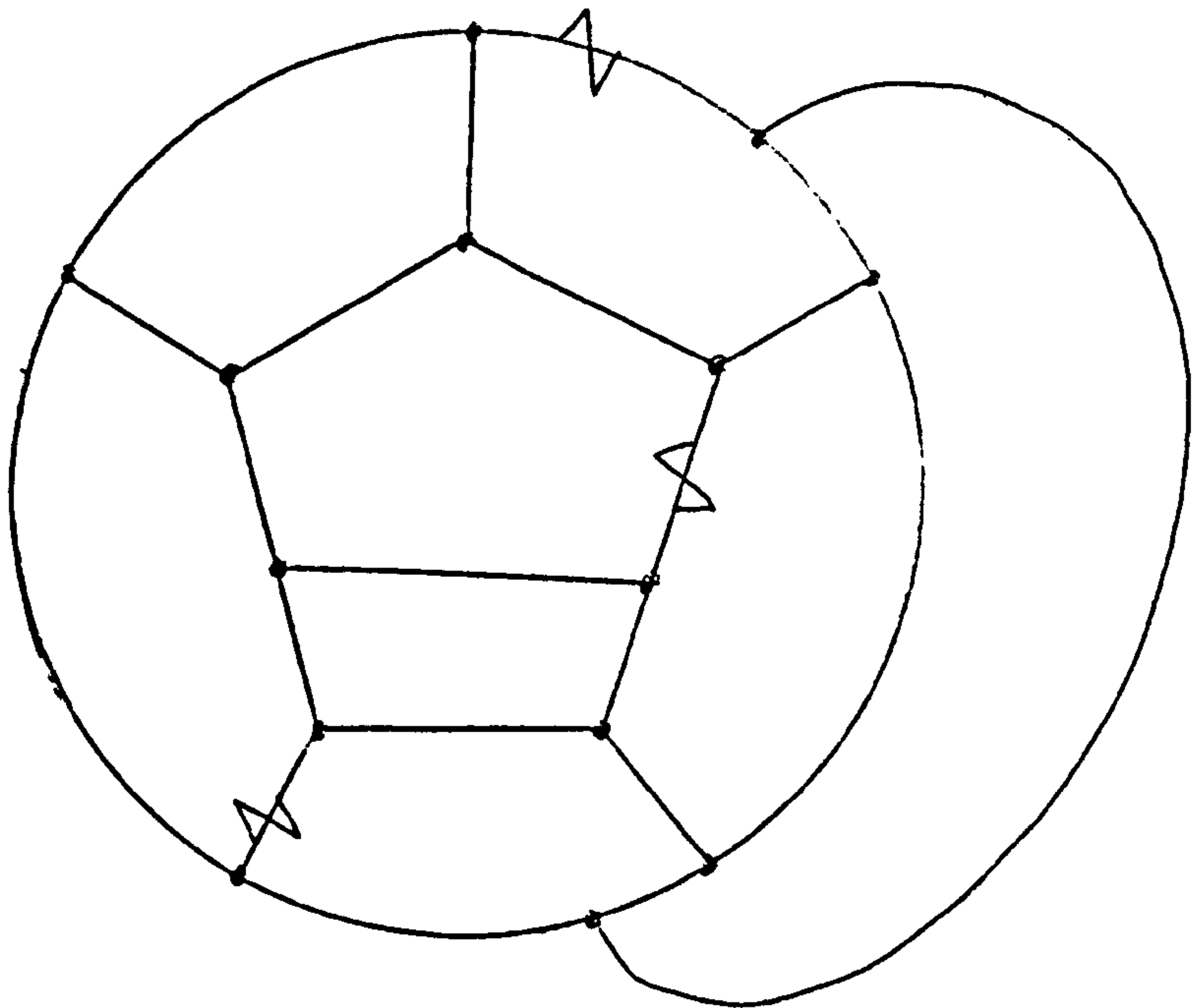


Fig 8.5

Proof

Define $H_0 = E(G)$; this clearly obeys condition (a). Suppose we have defined H_1 obeying condition (a) but such that, for some vertex v ,

$$|H_1 \cap E_v| > \frac{1}{2}|E_v|.$$

Then define H_{i+1} to be the symmetric difference between H_i and E_v . Since any circuit intersects E_v in 0 or 2 edges, H_{i+1} still obeys condition (a). But $|H_{i+1}| < |H_i|$, so eventually, since G is finite, we arrive at $H_t = H$ for which condition (b) holds (see Fig 8.6). ■

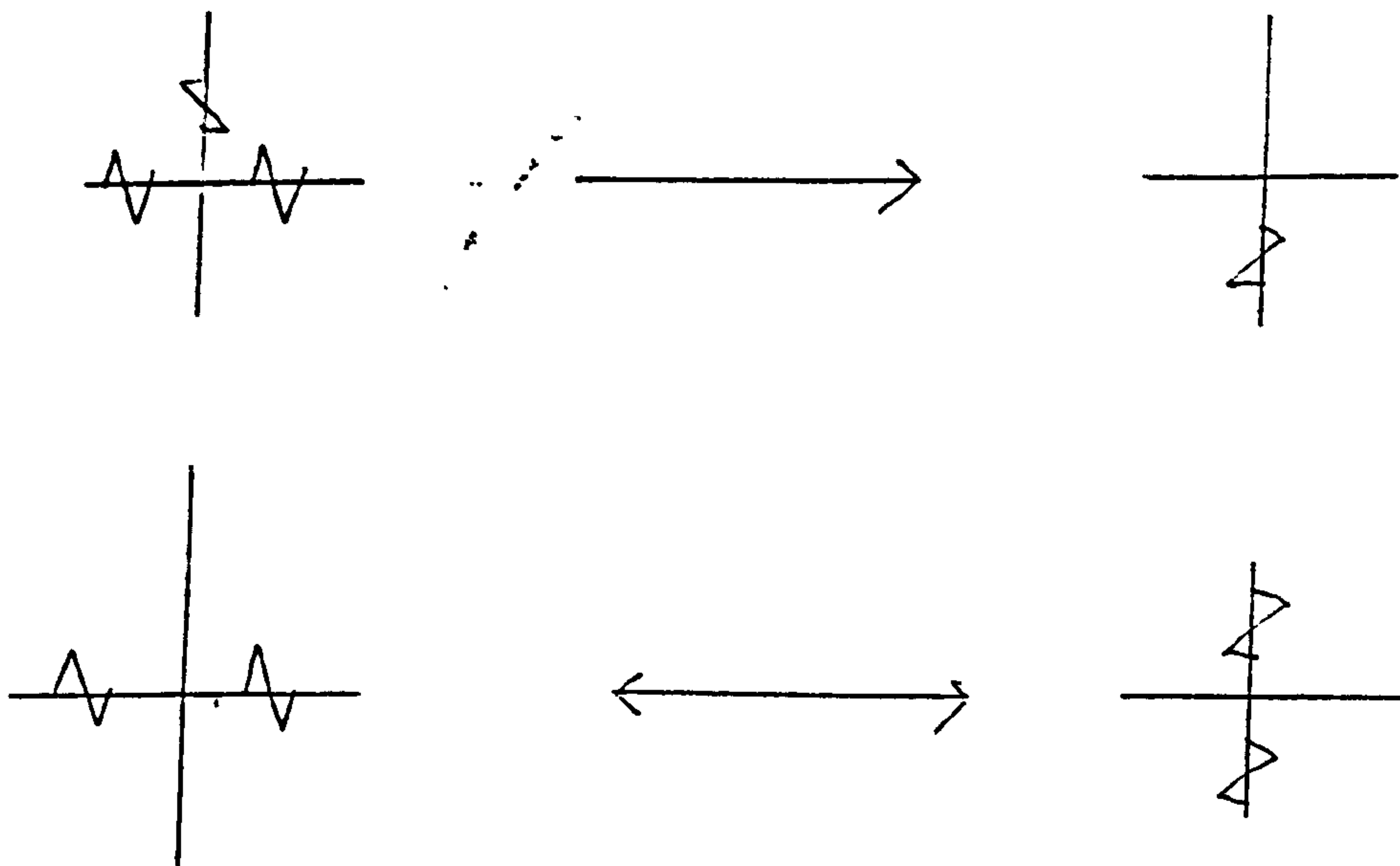


Fig 8.6

Thus Theorem 7.2 is really a corollary of this theorem. The case when Q is quartic goes as follows.

Corollary 8.1

Given a plane quartic graph (V, E) , it is possible to find a subset H of E so that (a) and (b) hold:

(a) For each odd circuit S , $|S \cap H| \equiv |S| \pmod{2}$

(b) At most two edges at each vertex are in S .

GLOSSARY

An *alternate labelling* in a quartic graph is a labelling of the 4 angles about each vertex with one or other of the colouring schemes $(0,1,0,1)$, $(1,0,1,0)$.

An alternate labelling in a quartic graph is said to be an *Alternate Angle labelling* if each 0-loop and each 1-loop is of even length.

The *chromatic number* [resp. *index*] of graph G is the least number of colours required in a vertex [resp. edge] colouring of G , and is denoted by $X(G)$ [resp. $X'(G)$].

A *circuit* in a graph is a sequence of vertices, each adjacent to the next, and uniquely so, the last being adjacent to the first.

Let G be a plane graph each of whose vertices is of degree 2 or 3. Consider two labellings $X: A(G) \rightarrow \{a, a^2\}$ and $Y: E_s(G) \rightarrow \{d, b\}$. For each face f of G , we define $P_f(X, Y)$ to be the product of the labels of the angles and edge sides incident with f , taken in cyclic order.

Let G be a P.C.B. graph with labellings

$$X: V(G) \rightarrow \{a, a^2\} \quad \text{and} \quad Y: E(G) \rightarrow \{d, b\}.$$

Then X and Y are *compatible* if $P_f(X, Y) = d$ for each face of G .

A *crossing point* in a plane map is a point where a graph is mapped many - one into the plane. Where three or more edges cross it is understood that each edge involved crosses every other edge just once.

"Cubic" is synonymous with "trivalent" and denotes a graph of degree 3.

A *cycle* in a planar graph is a simple closed curve that crosses a set of edges in a simple way --- i.e. each edge is crossed at an interior point at most once.

An *even component generator* or *E.C.G.* is a subgraph $\{V(G), S\}$, all of whose components are of even order.

A *dangling edge* in a hypograph is an edge having valency one at one end and valency greater than one at the other. (See also *hypograph*)

Two angles in a plane quartic map are *edgewise adjacent* if they are incident to a common edge but lie at opposite ends of that edge.

Two edges are said to be *geometrically adjacent* if they are incident with a common angle.

A pair of angles are called *geometrically alternate* if they lie on opposite sides of, and at opposing ends of, some edge in a planar graph.

A *graph* G is a vertex set $V(G)$ together with an edge set $E(G)$ of unordered pairs (u, v) of elements of $V(G)$.

A *Guthrie* colouring of a map is a face 4 colouring.

Heawood's congruences are those sums, modulo 3 of the Heawood labels of either vertices or angles about each face. We normally require such a sum to be zero modulo 3 (*).

A *Heawood labelling* is a function $f:V(G) \rightarrow \{1,2\}$ obeying (*) above.

A *Hypergraph* is a vertex set $V(G)$ together with a set of unordered subsets of 2 or more elements of $V(G)$ (and at least one subset of size greater than 2).

A *hypograph* may be formally defined as a vertex set $V(G)$ together with a list $E(G)$ of subsets of $V(G)$, each of size 1 or 2 (and at least one of size 1), the edges. Edges of size 1 are dangling edges while those of size 2 are normal edges. We do not allow multiple edges in a hypograph. Thus a *hypograph* is a graph containing at least one vertex of valency less than three. A vertex of valency one may also be viewed as the termination of an edge after half of it has been removed. The notion of *face* in a plane hypograph is taken to be any circuit that does not contain any other circuit. A hypograph is said to be *regular* if all the larger vertices (valency > 2) have the same valency. Thus a hypograph is just an ordinary graph with trees added. Because the structure of circuits is the same, colouring properties are unchanged by the addition or removal of trees. Finally it will be seen that each dangling edge can be drawn in the plane in two different ways.

By K_4 or K we denote that group $\{ a, b, c, d : x+x = d, x+d = x \}$

A *Kleinian* labelling of the edges of a graph is an assignment of elements of K the Klein 4 -- group to its edges so that two conditions hold :

- (1) Geometrically adjacent edges take different labels.
- (2) The sum of edge labels around each vertex is d , the identity in K .

We say that a plane graph is *Kleinian* if it has a Kleinian labelling.

A *loop* is a sequence of distinct angles a_i , such that

- (1) successive angles are edgewise adjacent, including the first and last.
- (2) an edge appears at most once in (1).

An *i-loop* ($i=0,1$) in a plane quartic graph, with an alternate labelling, is a loop all of whose angles are labelled i .

Let G be a plane graph. Then the *Medial graph* G_S is that graph whose vertices are in 1-1 correspondence with the edges of G , two vertices of G_S being adjacent if and only if the corresponding edges in G are geometrically adjacent.

Let G be as above. Then *v-faces* and *f-faces* in G_S are faces corresponding to the vertices and faces of G resp.

A *normal* edge in a hypograph is any edge other than a dangling edge. Thus a normal edge might become a dangling edge after removal of one or more dangling edges. (See also *dangling edge*, *hypograph*)

An *odd circuit sample* is a subset of the edge set in which at least one edge in each odd circuit is included, and at least one excluded.

Let G be any graph each of whose components is of even order. Then an *odd cut* of G is a minimal set of edges C of G whose removal would result in a graph having two components of odd order.

An *odd cut intersector* of a graph G is a set of edges having non zero intersection with each odd cut of G .

An *odd cut sample* is a subset of the edge set in which at least one edge of each odd cutset is included, and at least one excluded.

The *order* of a face of a map is the number of edges with which it is incident.

A *face-even* map is one in which each face is of even order.

A *vertex-even* graph is one in which each vertex is of even order.

A *triple* map is one in which the order of each face is a multiple of 3.

A *p.c.b* map is one that is at once cubic, bridgeless and planar. These maps are the critical cases for Guthrie colorability.

A *pentagram* is a drawing of a 5-circuit in such a way that each edge crosses both the edges to which it is not adjacent. If a pentagram P is

part of a drawing in the plane of a cubic graph, then its *surrounding regions* are those regions each of whose boundaries contain 2 vertices of P.

"*Quartic*" is synonymous with "4 -- valent" and denotes a graph of degree 4 everywhere.

A *region* in a graph drawn in the plane with a finite number of crossing points is a connected component of the complement of the drawing.

A *region 4 colouring* of a hypograph with crossing edges is a Guthrie colouring of its regions with one of two schemes of colouring of the regions around a crossing point :-

Either (a) all four colours appear

Or (b) the four regions take 2 out of the 4 colours, so that opposing regions take the same colour.

By S_3 we mean that group of order six with d the identity defined by

$$S_3 = \{ a, b : a^3 = b^2 = d; ab = ba^2 . \}$$

The *sense of rotation* of the Guthrie colouring of the three faces about a cubic vertex v is the same as that of the Tait colouring of the three edges incident at v to which the Guthrie colours correspond.

A *snark* is a connected bridgeless graph with edge - chromatic number 4.

(i.e. of class 2.)

A *Tait* colouring in a cubic map is a 3 -- colouring of its edge set.

A plane map is said to be *triple* if the order of each face is divisible by three.

Given an alternate angle labelling in a plane quartic graph G . Let F be a particular 0 or 1 loop. Then by the *turning label* T of angles of F , we mean labels L or R according as a left or right turn is made at the angle in traversing F in counterclockwise sense.

A snark is *prime* if it has no proper subgraph which is also a snark.

A spanning set of disjoint even circuits is called *uniform* if any circuit in the graph intersects the uniform set in an even number of edges.

A cubic graph is *Watkins* colourable if one can 4 - colour its vertices so that the neighbours of any vertex v have three colours, different from each other, and different from that of v itself.

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ABSTRACT

The Four Colour Theorem, Guthrie's proven conjecture, is a simple statement about the colourability of plane maps. In one go it unites the set of Tait colourable cubic graphs with the topological property of planarity. If anything one should be amazed at this Theorem, as one might expect at least sporadic non-colourable graphs to occur as one examines arbitrarily large, or dense graphs.

There are many propositions equivalent to Guthrie's conjecture. The most interesting of these are those most nearly connected with the original proposition rather than those much less tightly coupled to it. These other results are validated by the Four Colour Theorem, rather than the other way round.

To the list of Guthrie's, Tait's and Heawood's conjectures, we add some new labelling propositions in various abstract groups of size from two to six elements. In Z_2 we have the Y-property that does not coincide with any other known property, yet confers four colourability. There are presented some other properties of cubic graphs that are ordered by implication, Tait colourability appearing centrally.

We examine colourings of the regions of a graph nearly drawn in the plane. There arise two families of snark: the odd and the even cases. Heawood's Theorem is extended to edge and angle labellings in groups Z_2 and Z_3 , and then reduced to a simple formulation in S_3 .

Next we label edges of 3- and 4-regular graphs with elements of the Klein 4-group. Finally we extract a result in binary edge labelling that invites further investigation.

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Technical Report No. 263
November, 1976

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